MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

VOL. LXXXIX. NOVEMBER 9, 1928. NO. 1

Rev. T. E. R. PHILLIPS, PRESIDENT, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

Mervyn Archdall Ellison, The Observatory, Armagh, and Trinity College, Dublin ;

Michel S. Kovalenko, Ph.D., Assistant Professor of Mathematics and Astronomy, Swarthmore College, and Sproul Observatory, Swarthmore, Pa., U.S.A. ; and

Arthur Douglas Merriman, M.A., B.Sc., A.I.Mech.E., 13 Grosvenor Drive, Whitley Bay, Northumberland.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Charles Edward Barns, The Pacific Science Press, Morgan Hill, California, U.S.A. (proposed by R. G. Aitken) ;

Alfred Dillon Bell, Waihemo, Palmerston South, Otago, New Zealand (proposed by C. E. Adams) ;

Hugh G. Boutell, 3723 Jocelyn Street, Washington, D.C., U.S.A. (proposed by James Stokley) ;

Boris P. Gerasimovič, Harvard College Observatory, Cambridge, Mass., U.S.A. (proposed by Harlow Shapley) ;

Percy William Glover, c/o Boys' High School, New Plymouth, New Zealand (proposed by C. E. Adams) ;

Sydney Goldstein, Ph.D., St. John's College, Cambridge (proposed by Harold Jeffreys) ;

Yusuke Hagihara, Assistant Professor of Astronomy in the Imperial University of Tokyo, Astronomical Observatory, Tokyo, Japan (proposed by H. F. Baker) ;

Leon Hufnagel, Ph.D., The Observatory, Lund, Sweden, and Harvard College Observatory, Cambridge, Mass., U.S.A. (proposed by H. Dingle) ;

Ronald Alexander McIntosh, 1 Melford Street, Ponsonby, Auckland, New Zealand (proposed by L. J. Comrie) ;

Thomas Pattinson Marshall, Lingfield, 74 Park Avenue, Hull, Yorkshire (proposed by Charles H. Adamson) ;

Jeffrey Charles Percy Miller, B.A., Trinity College, Cambridge (proposed by A. S. Eddington) ;

Joseph Pearce, 72 Barclay Road, Leytonstone, London, E. 11 (proposed by A. Saywell) ;

Ralph Ernest Sadler, A.M.I.C.E., The Mansion, Ashbourne, Derbyshire (proposed by J. T. W. Claridge) ;

Paul Slavenas, Kaunas University, Lithuania (proposed by F. Schlesinger) ; and

Otto Struve, Ph.D., Assistant Professor in the University of Chicago, Yerkes Observatory, Williams Bay, Wisconsin, U.S.A. (proposed by E. B. Frost).

Two hundred and forty-two presents were announced as having been received since the last meeting, including amongst others :—

Cape of Good Hope, Royal Observatory : First Cape Catalogue of stars for the equinox 1925·0.

———————————— : Magnitudes of stars contained in the Cape Zone Catalogue of 20,843 stars.

Greenwich, Royal Observatory : Declinations of stars derived from observations of transits in the prime vertical with the altazimuth in the years 1923–26.

Sir J. H. Jeans : Eos, or the wider aspects of cosmogony.

Melbourne Observatory : Astrographic Catalogue, Vol. II.

Mount Wilson Observatory : Papers, Vol. III. Revision of Rowlands' Preliminary Table of Solar Spectrum Wave-Lengths, with an extension to the present limit of the infra-red.

Oxford University Observatory : Miscellaneous Papers, Vol. VII., 1921–27. Edited by H. H. Turner.

Isaac Roberts : Atlas of 52 regions, a guide to Herschel's fields (presented by Mrs. Isaac Roberts).

Vienna, Österreichische Kommission für die internationale Erdmessung : Protokolle über die Sitzungen 1922–1927.

Vienna, Universitäts-Sternwarte : Annalen, Band 24. Katalog von 26,060 sternen.

Framed photograph of the Royal Astronomical Society's rooms at Somerset House (presented by Mr J. G. Hawke).

Celestial Photographs.

The following have been added to the list of Celestial Photographs reproduced by the Society for sale to the Fellows :—

R.A.S. Ref. No.

402. Andromeda Nebula (southern portion), 100-in. reflector, Mount Wilson, Hubble.
403. Prominences and Flocculi, Calcium (H), 1913 March 7 (Sunspot minimum), Evershed.
404. Prominences and Flocculi, Calcium (H), 1918 Jan. 11 (Sunspot maximum), Evershed.
405. Mars, 1926 Oct. 14, 15, 16, 18. Exposures in V and U-V light, showing cloud. Wright, Lick Observatory.
406. Mars, 1926 Oct. 16. Exposures in U-V, G, Y, R, I-R, and V light. Wright, Lick Observatory.
407. Mars, 1926 Nov. 2, 3. Exposures Nov. 2, 3 in I-R and Nov. 2 in U-V light. Wright, Lick Observatory.
408. Mars, 1926 Nov. 2. Exposures in U-V and I-R light. Wright, Lick Observatory.
409. Saturn, 1927 July 13, 14. Exposures in U-V, V, Y, and R light. Wright, Lick Observatory.
410. Jupiter, 1927 Sept. 19. Exposures in U-V, V, G, Y, R, and I-R light. Wright, Lick Observatory.
411. Jupiter, 1927 Oct. 2. Exposures in U-V, V, G, Y, R, and I-R light. Wright, Lick Observatory.
412. Venus, 1927 Nov. 23, Dec. 8. Exposures in U-V and I-R light. Wright, Lick Observatory.

Erratum.

No. 399. Jupiter (Wright, Lick Observatory). The date of this photograph is 1927, *not* 1926 as previously stated.

The Theoretical Contours of Absorption Lines in Stellar Atmospheres.

By E. A. Milne.

1. "*The Number of Atoms per Square Centimetre above the Photosphere.*"—The recent important work of Unsöld,* of Miss Payne,† and of Adams and Russell ‡ has been largely concerned with the determination, by line-contours and line-intensities, of the number of atoms of a given kind or given state of excitation and ionization per square centimetre column above the photosphere of the sun or of any star. If the photosphere of a star were a definite geometrical surface, the phrase "amount of material above the photosphere" would have an unambiguous meaning; the atmospheric region would be a definite entity, just as

* *Zeits. für Phys.*, **44**, 793, 1927; **46**, 765, 1928.
† *Proc. Nat. Acad. Sci.*, **14**, 399, 1928.
‡ *Astrophys. Journ.*, **68**, 9, 1928.

the earth's atmosphere is a definite entity. Actually the layers contributing the general opacity must merge into the layers contributing the line opacity; reversing layer merges into photosphere. It is clear (and the idea has already been used by Adams and Russell in the paper cited) that the "amount of material above the photosphere," as found by various methods, must mean the amount of material above some particular optical depth $\tau = \tau_0$. This depth $\tau = \tau_0$ has the property that the actual atmosphere behaves in the same way as a model atmosphere in which the matter above $\tau = \tau_0$ is sliced off, rendered transparent except in its absorption lines, and put in front of a suitable radiating surface.

For a purpose which will be described in a later paper, the present investigation was carried out with the object of determining the value of τ_0, to which Unsöld's and Miss Payne's work refers. Their determinations of N, the number of atoms of a given species above the photosphere per cm.^{2}, depend on an application of a famous formula of Schuster's * for the decrease in intensity caused by an atmosphere of finite thickness in front of a definite radiating surface. The radiating surface is taken to radiate with an intensity equal to that of the continuous spectrum between the absorption lines—the photospheric intensity. The atmosphere is thus supposed to be transparent save in the line under discussion. But to determine the quantity τ_0 we must take account of the simultaneous setting-in of both general opacity and line opacity; we must attribute some degree of general opacity to the atmosphere. When this is done, as the following investigation will show, it is found that the intensity at any given frequency in an absorption line can no longer be represented by a formula of Schuster's type. It must be borne in mind that the contour of an absorption line in a stellar spectrum depends upon two factors: (*a*) the variation of intrinsic atomic opacity in the neighbourhood of the line-frequency, (*b*) the structure of the stellar atmosphere. It will appear that whatever (*a*) may be, the factor (*b*) (when account is taken of simultaneous general opacity and line opacity) causes the absorption-line contour to be slightly above that given by Schuster's formula near the centre of the line and slightly below it in the outer wings. Many of Unsöld's observed line-contours show an effect of this kind. A fit may be made at any desired point. This then determines the value of τ_0, to which the deduced number of atoms N applies. We shall find that if a fit is made where the ratio (line-intensity/continuous background) has the value $\frac{1}{2}$ (as seems to correspond to Unsöld's fits), then τ_0 is about $\frac{1}{3}$.

2. *The Fundamental Equations.*—I have shown in a recent paper † that when collisions are taken into account, the equation of transfer in a line-frequency takes the form

$$\cos\theta \frac{dI_\nu(\theta)}{s_\nu \rho dx} = -I_\nu(\theta) + \frac{\int I_\nu(\theta)(d\omega/4\pi) + \eta B_\nu}{1+\eta} \quad . \quad (1)$$

* Actually Unsöld used a modification of this formula due to the present writer; the modification takes account of the variation of intensity with angle of emergence.

† *M.N.*, **88**, 493, 1928.

where $I_\nu(\theta)$ is the intensity in a direction θ with x, ρ is the density, B_ν is given by

$$B_\nu = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{h\nu/kT} - 1},$$

s_ν is the monochromatic absorption coefficient of the material for frequency ν, and η is proportional to ρ/s_ν. The value of η thus increases from 0 indefinitely as ρ increases. Between the lines, in the continuous background, the equation of transfer (in the absence of general scattering) is simply

$$\cos\theta \frac{dI_\nu(\theta)}{\rho\kappa dx} = -I_\nu(\theta) + B_\nu \quad . \quad . \quad . \quad (2)$$

where κ is the coefficient of general opacity. Though (1) has only been established for a line-frequency, it tends to the form (2) when we replace s_ν by $\bar{\kappa}$ in η; for then, s_ν being large compared with $\bar{\kappa}$, η becomes large except for very small densities. Thus (1) represents the general behaviour for all ν.

Consider now the equation of transfer built up from purely macroscopic considerations for a material of *scattering* coefficient s_ν and true absorption coefficient κ_ν. We find

$$\cos\theta \frac{dI_\nu(\theta)}{\rho dx} = -(\kappa_\nu + s_\nu)I_\nu(\theta) + s_\nu \int I_\nu(\theta)(d\omega/4\pi) + \kappa_\nu B_\nu \quad . \quad (3)$$

This is precisely of the form (1), with $\eta = \kappa_\nu/s_\nu$, provided (1) is corrected for general absorption, specifically neglected in deriving it. Thus the true behaviour of the material, when account is taken of monochromatic absorption and re-emission modified by collisions, is mimicked by a material which simply scatters and absorbs in the way defined by macroscopic considerations. It is convenient to handle equation (3) rather than (1), as it applies as it stands to the adjacent continuous spectrum as well as to the line-frequency on introducing general scattering and giving appropriate values to s_ν and κ_ν.

It is possible to solve equation (1) or (3) with $\eta = \kappa_\nu/s_\nu \propto \rho$ by the use of appropriate transcendental functions. In a former paper I have solved (1) by fitting a region in which $\eta = 0$ on to a region in which $\eta \to \infty$. In the present case, since the point of the investigation is to consider the simultaneous setting-in of general and selective opacity, it is convenient to take η = constant. Though η ranges from 0 to practically ∞, this assumption is not so bad as it seems, for $1/(1+\eta)$ only ranges from 1 to 0 and $\eta/(1+\eta)$ only from 0 to 1. We are simply assuming that a mean value of η exists from which the emergent radiation can be calculated as though η were constant. Our conclusions relate to the corresponding mean value of κ_ν/s_ν.*

Let $\bar{\kappa}$ and $\bar{s}$ be the values assumed by κ_ν and s_ν in the region free from absorption lines; we assume them independent of ν, for simplicity, but most of the analysis holds whatever functions they are of ν, pro-

* In the sequel we take $\kappa_\nu = \bar{\kappa}$, so that the mean value of η is $\bar{\kappa}/s_\nu$.

vided they change but slowly in the neighbourhood of a line. In our immediate applications we shall take $\bar{s} = 0$, but for an investigation contained in the appendix the more general case is required.

Equation (3) can be dealt with taking account of the variation of I_ν with θ. The resulting formulæ are complicated, and, moreover, are only of application in the case of the sun. We shall therefore average over θ. Our formulæ will then refer to the light from a star or to the integrated light from the sun's disc taken as a whole.

Let I_ν be the mean value of $I_\nu(\theta)$ for $0 \leqslant \theta < \frac{1}{2}\pi$; and I'_ν the mean value of $I_\nu(\theta)$ for $\frac{1}{2}\pi < \theta \leqslant \pi$. Measuring x inwards and θ from the negative x-direction, we obtain on multiplying (3) by $d\omega$ and integrating separately over the two hemispheres,

$$\tfrac{1}{2}\frac{dI_\nu}{\rho dx} = (\kappa_\nu + s_\nu)I_\nu - \kappa_\nu B_\nu - \tfrac{1}{2}s_\nu(I_\nu + I'_\nu) \quad . \quad . \quad (4)$$

$$\tfrac{1}{2}\frac{dI'_\nu}{\rho dx} = -(\kappa_\nu + s_\nu)I'_\nu + \kappa_\nu B_\nu + \tfrac{1}{2}s_\nu(I_\nu + I'_\nu) \quad . \quad . \quad (5)$$

Write

$$2(\bar{\kappa} + \bar{s})\rho dx = dt \quad . \quad . \quad . \quad . \quad (6)$$

so that t denotes twice the optical thickness. Write also

$$\frac{\kappa_\nu + s_\nu}{\bar{\kappa} + \bar{s}} = n_\nu \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{\kappa_\nu}{\kappa_\nu + s_\nu} = \xi_\nu^2 \quad . \quad . \quad . \quad . \quad (8)$$

Adding and subtracting (4) and (5) we find

$$\frac{d}{dt}(I_\nu + I'_\nu) = n_\nu(I_\nu - I'_\nu) \quad . \quad . \quad . \quad . \quad (9)$$

$$\frac{d}{dt}(I_\nu - I'_\nu) = \xi_\nu^2 n_\nu(I_\nu + I'_\nu - 2B_\nu) \quad . \quad . \quad (10)$$

We now make the further assumption that n_ν may be taken as being the same at all depths, or may be replaced by a suitable mean value.

The solution of (9) and (10) with the appropriate boundary conditions at the boundary of the star and in the far interior I have already given (with a trifling change of notation) in another paper.* We are only interested in the emergent radiation. The result gives for the emergent intensity,

$$I_\nu(0) = \frac{2\xi_\nu}{1 + \xi_\nu}\int_0^\infty B_\nu(t)e^{-n_\nu\xi_\nu t}n_\nu\xi_\nu dt \quad . \quad . \quad (11)$$

where now $I_\nu(0)$ denotes the average intensity averaged over the stellar disc.

The integral in (11) can be evaluated once the temperature distribu-

* *Phil. Trans. Roy. Soc.*, **223**, A, 240, 1922.

tion is known. This is fixed by imposing the condition of radiative equilibrium for the integrated radiation of all frequencies. Neglecting the absorption lines for this purpose, (9) and (10) become on integration with respect to ν,

$$\frac{d}{dt}(\mathrm{I}+\mathrm{I}') = \mathrm{I}-\mathrm{I}' \quad . \quad . \quad . \quad . \quad (12)$$

$$\frac{d}{dt}(\mathrm{I}-\mathrm{I}') = 0 \quad . \quad . \quad . \quad . \quad . \quad (13)$$

the omission of suffixes ν denoting integrated radiation, of all frequencies. Hence

$$\mathrm{I}-\mathrm{I}' = \text{const.} = \mathrm{F},$$

say, and

$$\mathrm{I}+\mathrm{I}' = \mathrm{F}(1+t),$$

since $\mathrm{I} = \mathrm{F}$, $\mathrm{I}' = 0$ at $t = 0$. The temperature is given by

$$\mathrm{I}+\mathrm{I}' = 2\mathrm{B},$$

where

$$\mathrm{B} = (\sigma/\pi)\mathrm{T}^4,$$

by the condition of radiative equilibrium.* Hence

$$\mathrm{B} = \tfrac{1}{2}\mathrm{F}(1+t) \quad . \quad . \quad . \quad . \quad (14)$$

$$\mathrm{T}^4 = \mathrm{T}_0{}^4(1+t) \quad . \quad . \quad . \quad . \quad (15)$$

It is known, of course, that a more accurate approximation to radiative equilibrium is $\mathrm{B} = \tfrac{1}{2}\mathrm{F}(1+\tfrac{3}{2}\tau) = \tfrac{1}{2}\mathrm{F}(1+\tfrac{3}{4}t)$, but it is important to use (14) in connection with (11), since it depends on a consistent set of approximations. To verify this, integrate (11) over all frequencies. The result is

$$\mathrm{F} = \frac{2\xi}{1+\xi}\int_0^\infty \mathrm{B}(t)e^{-\xi t}\xi dt \quad . \quad . \quad . \quad (16)$$

a result which should be true for all ξ. It will be found that (14) satisfies this.

Now expand $\mathrm{B}_\nu(t)$ to two terms by Taylor's theorem. Writing it

$$\mathrm{B}_\nu(t) = a_\nu + b_\nu t \quad . \quad . \quad . \quad . \quad (17)$$

we have

$$\frac{b_\nu}{a_\nu} = \left[\frac{1}{\mathrm{B}_\nu}\frac{d\mathrm{B}_\nu}{dt}\right]_{t=0} = \frac{1}{4}\frac{h\nu}{k\mathrm{T}_0}\cdot\frac{1}{1-e^{-h\nu/k\mathrm{T}_0}} \quad . \quad . \quad (18)$$

Higher terms in (11) will not contribute much, as when they become appreciable the factor $e^{-n_\nu\xi_\nu t}$ is very small. To take account of just two terms gets the spirit of the actual state of things, as it definitely introduces the effect of a temperature gradient, even if not quite the right gradient. It is readily found that for the visible region and stellar temperatures, b_ν/a_ν is of the order of unity. This is the principal fact which concerns us.

* σ is Stefan's constant.

We can now integrate (11), and we find

$$I_\nu(0) = \frac{2\xi_\nu}{1+\xi_\nu}\left(a_\nu + \frac{b_\nu}{n_\nu \xi_\nu}\right) \quad . \quad . \quad . \quad (19)$$

3. *Application to the Contour of an Absorption Line.*—Result (19) holds for all frequencies. We shall now assume that general scattering is negligible,* so that $\bar{s} = 0$. We shall further assume that true absorption is the same inside the line as outside it, *i.e.* $\kappa_\nu = \bar{\kappa}$. Then in the vicinity of the line we have

$$\xi_\nu^2 = \frac{\bar{\kappa}}{\bar{\kappa} + s_\nu}, \qquad n_\nu = \frac{\bar{\kappa} + s_\nu}{\bar{\kappa}} \quad . \quad . \quad . \quad (20)$$

In the neighbouring continuous spectrum we have $\kappa_\nu = \bar{\kappa}$, $s_\nu = \bar{s} = 0$, and so $\xi_\nu = 1$, $n_\nu = 1$. Introducing in (19) we have

$$I_\nu(0) = \frac{2\xi_\nu}{1+\xi_\nu}(a_\nu + b_\nu \xi_\nu) \quad . \quad . \quad . \quad (21)$$

$$i_\nu(0) = a_\nu + b_\nu \quad . \quad . \quad . \quad . \quad (22)$$

where $i_\nu(0)$ refers to the continuous spectrum adjacent to the line, for which a_ν and b_ν have the same values as in the line. Hence the intensity-ratio r is

$$r = \frac{I_\nu(0)}{i_\nu(0)} = \frac{2\xi_\nu}{1+\xi_\nu}\,\frac{a_\nu + b_\nu \xi_\nu}{a_\nu + b_\nu} \quad . \quad . \quad . \quad (23)$$

Formula (23) gives the intensity-ratio r as a function of ν, once s_ν is known as a function of ν; for then ξ_ν is known by (20). Formula (23) may be said to take account of (*a*) general absorption, (*b*) temperature radiation, (*c*) absorption and remission without change of frequency, which is equivalent to monochromatic radiative equilibrium in the line-frequency, allowing for some effect of collisions,† (*d*) a temperature gradient determined by the radiative equilibrium of the material as a whole. It appears to be free from the objection sometimes urged that this type of argument usually gives the thermodynamic result that the residual intensity never falls below that corresponding to the boundary temperature. On the contrary, by taking s_ν large enough we can make ξ_ν as small as we please, and so $I_\nu(0)$ as small as we please. The point is that we have not assumed thermodynamic equilibrium at any point; monochromatic scattering, when present, precludes thermodynamic equilibrium.

In discussing (23) it must be borne in mind that a_ν and b_ν change comparatively slowly with ν in the neighbourhood of an absorption line, whilst ξ_ν changes rapidly.

* This assumption is not essential to the arguments which follow; but, if we take $\bar{s}$ into account, we must also have for comparison a modification of Schuster's formula in which the general scattering is taken into account as well as the selective.

† The collisions when not elastic give rise to "temperature radiation," as follows from (1).

4. *Comparison with Schuster's Formula.*—We may as well show that our approximations are consistent with Schuster's formula by re-deriving it. For a purely scattering atmosphere in front of a radiating surface, we have equations (4) and (5) with $\kappa_\nu = 0$. We get

$$\frac{dI_\nu}{\rho dx} = s_\nu(I_\nu - I_\nu') = \frac{dI_\nu'}{\rho dx} \quad . \quad . \quad . \quad (24)$$

Hence

$$I_\nu - I_\nu' = \text{const.} = I_\nu(0) \quad . \quad . \quad . \quad (25)$$

Hence

$$I_\nu(x) = I_\nu(0)\int_0^x s_\nu \rho dx + \text{const.}$$

The constant is clearly $I_\nu(0)$. Hence

$$I_\nu(0) = \frac{I_\nu(x)}{1 + \int_0^x s_\nu \rho dx} \quad . \quad . \quad . \quad (26)$$

But $I_\nu(x)$ is the intensity at the background if this is at a distance x from the front boundary, and is therefore the photospheric intensity $i_\nu(0)$. Hence

$$\frac{I_\nu(0)}{i_\nu(0)} = \frac{1}{1 + \int_0^x s_\nu \rho dx} \quad . \quad . \quad . \quad (27)$$

which is Schuster's formula. To distinguish it from (23) we shall write the left-hand side as $I_\nu^*(0)/i_\nu^*(0)$ and denote it by r^*.

Our object is to compare the absorption-line contours given by (23) and (27) and to determine the optical depth, measured in general absorption, which reconciles (27) with (23).

From (20) we have

$$s_\nu = \bar{\kappa}\left(\frac{1}{\xi_\nu^2} - 1\right) \quad . \quad . \quad . \quad . \quad (28)$$

whence

$$r^* = \frac{I_\nu^*(0)}{i^*(0)} = \frac{\xi_\nu^2}{\xi_\nu^2 + (1 - \xi_\nu^2)\int_0^x \bar{\kappa}\rho dx} \quad . \quad . \quad (29)$$

But

$$\int_0^x \bar{\kappa}\rho dx = \tau_0,$$

the optical depth to the equivalent photospheric surface. Hence

$$r^* = \frac{I_\nu^*(0)}{i_\nu^*(0)} = \frac{\xi_\nu^2}{\xi_\nu^2 + (1 - \xi_\nu^2)\tau_0} \quad . \quad . \quad (30)$$

Formulæ (23) and (30) agree in giving an intensity-ratio which is unity when ξ_ν is 1 and which decreases as ξ_ν decreases. But they are not the same function of ξ_ν, and accordingly give different line-

contours. We proceed to investigate the relationship of the two curves.

We have from (23) and (30)

$$r - r^* = \frac{\xi_\nu(1-\xi_\nu)}{(1+\xi_\nu)(a_\nu+b_\nu)} \frac{2(a_\nu+b_\nu\xi_\nu)(1+\xi_\nu)\tau_0 - a_\nu\xi_\nu - b_\nu\xi_\nu(1+2\xi_\nu)}{\xi_\nu^2+\tau_0(1-\xi_\nu^2)} \quad (31)$$

Unsöld proceeds by fitting his observed curve to a Schuster curve, *i.e.* by choosing a value of $\int s_\nu \rho dx = \tau_0 s_\nu/\bar{\kappa}$, which gives agreement with the observed intensity-ratio. Let us suppose a fit is made where ξ_ν has the value ξ_ν', *i.e.* at a place where the intensity-ratio is

$$r' = \frac{2\xi_\nu'}{1+\xi_\nu'} \frac{a_\nu+b_\nu\xi_\nu'}{a_\nu+b_\nu} \quad . \quad . \quad . \quad . \quad (32)$$

Then we have from (31), on putting $r' - r'^* = 0$,

$$\tau_0 = \frac{a_\nu\xi_\nu' + b_\nu\xi_\nu'(1+2\xi_\nu')}{2(1+\xi_\nu')(a_\nu+b_\nu\xi_\nu')} \quad . \quad . \quad . \quad (33)$$

For brevity we put from now on

$$b_\nu/a_\nu = c_\nu \sim \frac{1}{4}\frac{h\nu}{kT_0} \quad . \quad . \quad . \quad . \quad (34)$$

Inserting for τ_0 from (33) in (31) we have

$$r - r^* = -\frac{2\xi_\nu(1-\xi_\nu)(\xi_\nu-\xi_\nu')}{(1+\xi_\nu)(1+c_\nu)} \cdot \frac{1+c_\nu(1+2\xi_\nu+2\xi_\nu'+\xi_\nu\xi_\nu')+c_\nu^2\xi_\nu\xi_\nu'}{\xi_\nu'(1+c_\nu+2c_\nu\xi_\nu')+\xi_\nu^2(2+\xi_\nu'+c_\nu\xi_\nu')} \quad (35)$$

From this formula we can see the general trend of the difference between the Schuster line-contour and the line-contour allowing for general absorption and the temperature gradient. For

when $\xi_\nu > \xi_\nu'$, *i.e.* when $s_\nu < s_\nu'$, we have $r < r^*$,
when $\xi_\nu < \xi_\nu'$, *i.e.* when $s_\nu > s_\nu'$, we have $r > r^*$.

If s_ν' refers to a point midway out along the wing of the contour, then $s_\nu > s_\nu'$ in the line centre and $s_\nu < s_\nu'$ in the outer wings. It follows that the contour calculated on the present assumptions falls below the Schuster contour in the outer wings of the line and rises above it in the centre.

To see the order of magnitude of the change consider the D lines of Na on the sun. For these $h\nu/k$T is about 4·83, so that c_ν is approximately unity. Suppose a fit with the Schuster ratio is made where the intensity-ratio r' is $\frac{1}{2}$. Then by (32) with $c_\nu = 1$ we have $\xi_\nu' = \frac{1}{2}$, and from (33) $\tau_0 = \frac{1}{3}$, whence

$$r - r^* = \frac{\xi_\nu(1-\xi_\nu)(1-2\xi_\nu)}{1+2\xi_\nu^2}.$$

We find the following values :—

$c_\nu = 1$; $\xi_\nu' = \frac{1}{2}$; $r' = \frac{1}{2}$; $\tau_0 = \frac{1}{3}$.

ξ_ν.	r.	$r - r^*$.
$\frac{1}{4}$	$\frac{1}{4}$	$+\frac{1}{12} = 0{\cdot}083$
$\frac{1}{2}$	$\frac{1}{2}$	0
$\frac{3}{4}$	$\frac{3}{4}$	$-\frac{3}{68} = -0{\cdot}044$

It will be seen that the differences are of the order of a few per cent. It will be an interesting question whether refined measures of line-contours will indicate an effect of this kind. Many, though not all, of Unsöld's contours do in fact show it.

5. *The Value of* τ_0.—With any given value of c_ν it is possible to solve (32) for ξ_ν' in terms of r' and then substitute in (33) to determine τ_0 as a function of r', our main object. The elimination of ξ_ν' is most quickly performed as follows :—

Equating both (23) and (30) to the common value r' of the intensity-ratio at the point of fit, we have

$$r' = \frac{\xi_\nu'^2}{\xi_\nu'^2 + \tau_0(1 - \xi_\nu'^2)} = \frac{2\xi_\nu'}{1 + \xi_\nu'} \frac{1 + c_\nu \xi_\nu'}{1 + c_\nu} \quad . \quad (36)$$

Writing $1/\xi_\nu' = y$, the first of these is equivalent to

$$y^2 = 1 + \frac{r'^{-1} - 1}{\tau_0} \quad . \quad . \quad . \quad . \quad (37)$$

and the second to

$$r'y = 1 + \frac{1 - c_\nu}{1 + c_\nu} \frac{y - 1}{y + 1} \quad . \quad . \quad . \quad (38)$$

Write

$$\frac{1 - c_\nu}{1 + c_\nu} = d_\nu.$$

Then (38) may be written

$$r'y^2 - 1 + d_\nu = y(d_\nu + 1 - r') \quad . \quad . \quad . \quad (39)$$

Eliminating y between this and (37), we find after some reduction

$$\frac{1}{\tau_0^2} - \frac{1}{\tau_0}\left[\frac{1 + r'}{r'} - d_\nu\left\{\frac{2(2r' - 1)}{r'(1 - r')} - \frac{d_\nu}{r'(1 - r')}\right\}\right] - \frac{4d_\nu}{1 - r'} = 0 \quad (40)$$

a quadratic for τ_0 ; we choose that root which tends to $r'/(1 + r')$ as $d_\nu \to 0$, *i.e.* the positive value for the surd in the solution for $1/\tau_0$.

We can see the order of magnitude of this root immediately. We have seen that c_ν is always of the order of magnitude of unity. For the D lines of Na on the sun, for example, $c_\nu = 1{\cdot}21$, and accordingly $d_\nu = -0{\cdot}095$. Now when $c_\nu = 1$ we have $d_\nu = 0$, and the root of (40) is

$$(\tau_0)_{c_\nu = 1} = \frac{r'}{1 + r'} \quad . \quad . \quad . \quad . \quad (41)$$

Suppose the fit of the observed line-contour with the Schuster contour is best where the intensity-ratio $r' = \frac{1}{2}$. This agrees with the curves given by Unsöld. Then we have at once

$$\tau_0 = \tfrac{1}{3}.$$

In general, the solution of (40) when d_ν is small is found to be approximately

$$\tau_0 = \frac{r'}{r'+1}\left(1 - \frac{2d_\nu}{(1+r')^2}\right) \quad . \quad . \quad . \quad (42)$$

It is convenient to recall that the value of d_ν is

$$d_\nu = \frac{1 - \frac{1}{4}\frac{h\nu}{kT_0}}{1 + \frac{1}{4}\frac{h\nu}{kT_0}} \quad . \quad . \quad . \quad . \quad (43)$$

(For the H and K lines of Ca^+ on the sun, $d_\nu = -\frac{1}{5}$ approximately.)

It may be observed that as $r' \to 1$,

$$\tau_0 \to \tfrac{1}{2} - \tfrac{1}{4}d_\nu \quad . \quad . \quad . \quad . \quad (44)$$

accurately for all values of d_ν. Thus as the fit is made more and more out towards the wings, τ_0 tends approximately to $\frac{1}{2}$.

We conclude that, subject to a small correction depending on the frequency of the line and the temperature of the star, we may adopt $\tau_0 = \frac{1}{3}$ as giving the photospheric depth to which the method of Unsöld applies. As the fit is made further out towards the wings than $r' = \frac{1}{2}$, we see down to deeper levels, but the value of τ_0 changes only slightly, namely, towards $\tau_0 = \frac{1}{2}$, subject to a similar small correction depending on ν and T. These results refer to observations made on the integrated radiation from the whole disc.

It may be pointed out that Eddington * has concluded that "the reversing layer should not be placed much lower than $\tau_0 = 0{\cdot}2$." Our result is in general agreement with his, and is indeed obtained by a similar method. Eddington, however, assumed a radiation intensity gradient in the continuous spectrum independent of frequency and temperature (the same as for the integrated radiation for all frequencies), whilst by the introduction of a_ν and b_ν and the associated c_ν and d_ν we have obtained the correction for ν and T_0.

A word may be said about the exact physical meaning of this depth $\tau = \tau_0$. The depth $\tau = \tau_0$ has the property that the material above $\tau = \tau_0$, if placed in front of a radiating background of the photospheric intensity and if transparent except in its absorption line, would produce for the frequency in question an absorption line of the same intensity-ratio r as it does in the actual stellar atmosphere. In the actual atmosphere the atoms beneath $\tau = \tau_0$ are contributing to some extent to the formation of the absorption line, but the whole effect is the same

* *The Internal Constitution of the Stars*, 1926, p. 342.

as could be produced by the atoms above $\tau = \tau_0$ acting alone in front of a radiating surface and deprived of general opacity.*

Though the Schuster contour and the contour calculated on the present assumptions fit fairly closely, the behaviour of the two contours as a function of s_ν is very different. When s_ν is fairly large compared with $\bar{\kappa}$, *i.e.* near the centre of the line, we have

$$r \sim \frac{a_\nu}{a_\nu + b_\nu}\left(\frac{s_\nu}{\bar{\kappa}}\right)^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (45)$$

$$r^* \sim s_\nu^{-1}\mathrm{M}^{-1} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

M being the total mass per cm.^{2}. Thus, for a given mass of scattering atmosphere, the intensity-ratio varies inversely as the scattering coefficient; but when simultaneous general absorption is taken into account the intensity-ratio varies inversely as the square root of the scattering coefficient. This has been pointed out by Eddington (*loc. cit.*), and is indeed shown in Schuster's classical work.

6. *Deductions from the Determination of* τ_0.—If M is the total mass above the optical level $\tau = \tau_0$, $\bar{\kappa}$ the mean general absorption coefficient, then

$$\bar{\kappa}\mathrm{M} = \tau_0 . \quad . \quad . \quad . \quad . \quad (47)$$

Suppose it is found by the method of Unsöld that there are N atoms of a particular kind, each of mass m, down to this level, and that this constituent forms a fraction ϵ by mass of the whole. Then

$$\mathrm{N}m = \epsilon\mathrm{M},$$

whence

$$\frac{\epsilon}{\bar{\kappa}} = \frac{\mathrm{N}m}{\tau_0} \quad . \quad . \quad . \quad . \quad (48)$$

Thus each line-contour determines a value of $\epsilon/\bar{\kappa}$. A knowledge of the absolute value of $\bar{\kappa}$ would thus determine ϵ, the absolute abundance of the element in question.

The line-contour may of course be analysed with respect to formula (23) directly instead of with respect to a Schuster formula. Each contour then determines at a given ν a particular intensity-ratio r', *i.e.* a value for ξ_ν', or for $s_\nu/\bar{\kappa}$. Now s_ν is the scattering coefficient of the material as a whole. If a_ν is the atomic scattering coefficient (a rapidly varying function of ν), then

$$s_\nu = \epsilon a_\nu/m.$$

Hence observation fixes $\epsilon a_\nu/m\bar{\kappa}$. Dividing out by the known value of a_ν/m for the particular frequency ν we have a value for $\epsilon/\bar{\kappa}$ as before.

We shall show in a later paper that the absolute value of $\bar{\kappa}$ can be

* The point is that when general opacity is taken into account, the deeper-lying atoms are screened by the higher-lying ones and do not have their full effect. The atoms between $\tau=0$ and $\tau=\tau_0$, if deprived of general opacity, have the same effect as the atoms down to depth infinity when endowed with general opacity.

found from the observed maxima of absorption lines in stellar spectra. In the meantime we can take a glance at the order of magnitude of $\bar{\kappa}$ indicated by Unsöld's results. For Na, Al, Ca on the sun he has found a value for each of the order of 4×10^{19} atoms cm.$^{-2}$. Their mean atomic weight is about 30. Assume these three elements together constitute 10 per cent. by mass of the whole atmosphere. Then we have

$$\bar{\kappa} = \frac{0{\cdot}10 \times 0{\cdot}33}{3 \times 30 \times 1{\cdot}66 \times 10^{-24} \times 4 \times 10^{19}} = 5{\cdot}5$$

for the mean mass absorption coefficient of the reversing layer. This estimate is much smaller than other estimates which have been made. Discussion of it is postponed to the following paper.

7. *Summary.*—(1) There are two main models of a stellar atmosphere. In one we have material, transparent except in line-frequencies, supposed placed in front of a radiating surface radiating with "photospheric" intensity. In the other we have no surface of demarcation between atmosphere and photosphere but a merging of the layers contributing the line opacity into those contributing the general opacity. Determinations of N, the number of atoms per cm.^{2} in a stellar atmosphere, presuppose the first model. The actual facts must correspond much more closely to the second model. Hence it is necessary to interpret values of N in terms of the second model. The present paper shows how to calculate for any point of a line-contour the optical depth $\tau = \tau_0$ for which the following property holds. Let the atoms in the second model existing above $\tau = \tau_0$ be placed in front of the radiating surface in the first model, and rendered transparent except in the line-frequency. Then the two models give the same intensity-ratio at the chosen point of the line-contour.

(2) For the first model, the contour of an absorption line may be calculated (given the atomic absorption coefficient as a function of ν) by means of a Schuster formula. For the second model a different formula holds, and the contour is slightly different. The theoretical contour for the second model (calculated on certain simplifying assumptions) rises above the Schuster contour in the line-centre and falls below it in the wings.

(3) A fit of the two contours may be made at any desired place on the line-contour. This fit corresponds to a fixation of N, "the number of atoms above the photosphere," as determined by the method of Unsöld, and to an evaluation of the level $\tau = \tau_0$, the base of the column of N atoms; τ_0 measures the *general* opacity down to this level (*i.e.* the opacity in the absence of an absorption line).

(4) Subject to a small correction for frequency and temperature, the value of τ_0 is $r/(r+1)$, where r is the intensity-ratio at the point of the line-contour where the fit is made. Unsöld's fits correspond to about $r = \frac{1}{2}$, which gives $\tau_0 = \frac{1}{3}$. As the fit is made further out to the wings, r tends to unity and τ_0 tends to $\frac{1}{2}$. These values of τ_0 refer to integrated radiation from the whole stellar disc.

(5) Each determination of N by the method of Unsöld determines in effect a value of $\epsilon/\bar{\kappa}$, where $\bar{\kappa}$ is the mean coefficient of general

absorption in the reversing layer and ϵ is the absolute abundance of the atoms in question by mass. (This assumes N properly corrected for ionization, the method for which is the subject of the following paper.)

(6) Assuming that Na, Al, and Ca form 10 per cent. by mass of the solar atmosphere, Unsöld's results give $\bar{\kappa} = 5{\cdot}5$ approximately.

Appendix.

Note on Schuster's Theory of dark and bright Lines in Stellar Spectra.

Formula (11) lends itself to a re-discussion by Schuster's method * of the occurrence of bright and dark lines in stellar spectra. The formula takes account of the complete depth of the photosphere as far as we can see into it, instead of involving the assumption of a photospheric radiating surface.

At the time of Schuster's paper the view was that s_ν is fairly continuous (changes slowly) throughout the spectrum, whilst κ_ν suffers a rapid increase (approximately a discontinuity) at a line-frequency; s_ν might be taken, for instance, to correspond to molecular or Rayleigh scattering and to be proportional to the fourth power of the frequency. The modern view is that κ_ν, the "true absorption" coefficient, is fairly continuous, whilst s_ν suffers an abrupt increase at a line-frequency; for quantum absorption and re-emission without change of frequency are formally equivalent to scattering. Let us trace out the consequences of the two views.

On either view we have $n_\nu = 1$ in the adjacent continuous spectrum. Write

$$\xi^2 = \frac{\bar{\kappa}}{\bar{\kappa} + \bar{s}} \quad . \quad . \quad . \quad . \quad (49)$$

which is the value of ξ_ν^2 in the continuous spectrum. Using approximation (17) in (11) we obtain (19), whence for the line-frequency

$$I_\nu(0) = \frac{2\xi_\nu}{1 + \xi_\nu}\left(a_\nu + \frac{b_\nu}{n_\nu \xi_\nu}\right) \quad . \quad . \quad . \quad (50)$$

and for the adjacent continuous spectrum

$$i_\nu(0) = \frac{2\xi}{1 + \xi}\left(a_\nu + \frac{b_\nu}{\xi}\right) \quad . \quad . \quad . \quad (51)$$

Hence the line will be dark or bright according as

$$\frac{\xi_\nu}{1 + \xi_\nu}\left(a_\nu + \frac{b_\nu}{n_\nu \xi_\nu}\right) \lessgtr \frac{\xi}{1 + \xi}\left(a_\nu + \frac{b_\nu}{\xi}\right)$$

or

$$\frac{\xi_\nu}{1 + \xi_\nu}\left(1 + \frac{c_\nu}{n_\nu \xi_\nu}\right) \lessgtr \frac{\xi}{1 + \xi}\left(1 + \frac{c_\nu}{\xi}\right) \quad . \quad . \quad (52)$$

* *Astrophys. Journ.*, **21**, 1, 1905.

(a) *The Older View.*—Here, in the line-frequency κ_ν is large and $s_\nu = \bar{s}$. Hence

$$\xi_\nu^2 = \frac{\kappa_\nu}{\kappa_\nu + \bar{s}}, \qquad n_\nu = \frac{\kappa_\nu + \bar{s}}{\bar{\kappa} + \bar{s}} \quad . \quad . \quad . \quad (53)$$

whence

$$n_\nu = \frac{1 - \xi^2}{1 - \xi_\nu^2} \quad . \quad . \quad . \quad . \quad (54)$$

The line is dark or bright according as

$$\frac{\xi_\nu}{1 + \xi_\nu} + c_\nu \frac{1 - \xi_\nu}{1 - \xi^2} \lessgtr \frac{\xi}{1 + \xi} + \frac{c_\nu}{1 + \xi},$$

i.e. according as

$$(\xi_\nu - \xi)\left[\frac{1}{1 + \xi_\nu} - \frac{c_\nu}{1 - \xi}\right] \lessgtr 0 \quad . \quad . \quad . \quad (55)$$

Now since $\kappa_\nu > \bar{\kappa}$, $\xi_\nu > \xi$. Hence the line is dark or bright according as

$$c_\nu \gtrless \frac{1 - \xi}{1 + \xi_\nu},$$

i.e. according as

$$\frac{1}{4}\frac{h\nu}{k\mathrm{T}_0} \gtrless \frac{1 - \left(\frac{\bar{\kappa}}{\bar{\kappa} + \bar{s}}\right)^{\frac{1}{2}}}{1 + \left(\frac{\kappa_\nu}{\kappa_\nu + \bar{s}}\right)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad (56)$$

Since κ_ν is large compared with $\bar{s}$, the denominator on the right-hand side is approximately 2.

We now notice that if $\bar{s} = 0$, the right-hand side is zero, and the upper inequality holds. *In the absence of general scattering all the lines are dark.* On the other hand, if $\bar{s} \neq 0$, the lower inequality will hold for sufficiently small values of ν. *In the presence of general scattering, lines of low frequency may be bright, but will tend to become dark with increasing frequency.* The conclusions in italics are precisely Schuster's results. We have thus derived Schuster's results without introducing a radiating background surface, and on the whole with simpler analysis. The conditions for a bright line are easily seen to be

$$\frac{h\nu}{k\mathrm{T}_0} < 2 \quad . \quad . \quad . \quad . \quad . \quad (57)$$

$$\frac{\bar{s}}{\bar{\kappa}} > \left(1 - \frac{1}{2}\frac{h\nu}{k\mathrm{T}_0}\right)^{-2} - 1 \quad . \quad . \quad . \quad (58)$$

For the red Balmer line $\lambda 6563$, at 15,000°, $h\nu/k\mathrm{T}_0$ is about 1·74, so that the first condition is just satisfied. In addition we then require $\bar{s}/\bar{\kappa} > 60$, which is scarcely likely to be realised. Physically, Schuster's explanation of bright lines demands a heavily scattering atmosphere to dim the background accompanied with a mechanism for furnishing energy to the line constituent by means of temperature radiation. It would seem impossible to maintain it.

(b) *The Modern View.*—The continuous spectrum is calculated as before. But now, in the line frequency we take s_ν large and $\kappa_\nu = \bar{\kappa}$. Thus now

$$\xi_\nu^2 = \frac{\bar{\kappa}}{\bar{\kappa} + s_\nu}, \qquad n_\nu = \frac{\bar{\kappa} + s_\nu}{\bar{\kappa} + \bar{s}} \quad . \quad . \quad . \quad (59)$$

Hence

$$n_\nu = \frac{\xi^2}{\xi_\nu^2} \quad . \quad . \quad . \quad . \quad (60)$$

which should be contrasted with (54). The line is dark or bright according as

$$\frac{\xi_\nu}{1 + \xi_\nu} + c_\nu \frac{\xi_\nu^2}{\xi^2(1 + \xi_\nu)} \gtrless \frac{\xi}{1 + \xi} + \frac{c_\nu}{1 + \xi},$$

i.e. according as

$$(\xi_\nu - \xi)\left[\frac{1}{(1 + \xi_\nu)(1 + \xi)} + c_\nu \frac{\xi_\nu + \xi + \xi_\nu \xi}{\xi^2(1 + \xi_\nu)(1 + \xi)}\right] \lessgtr 0 \quad . \quad (61)$$

Since $s_\nu > \bar{s}$, $\xi_\nu < \xi$, and the expression on the left-hand side is always essentially negative; hence the line is always dark whatever the temperature gradient or scattering coefficient. Monochromatic selective scattering * always leads to a dark line. This result has been reached by various paths by several writers, but it has seemed worth while to establish it from exactly the same formula as is used in Schuster's explanation of bright lines.

The whole theory of intensities in absorption lines derives from Schuster's pioneer paper of 1905. The above re-statement of Schuster's theory in modern terms may therefore be of interest at the present moment.

Ionization in Stellar Atmospheres. Part I. Generalised Saha Formulæ, Maximum Intensities, and the Determination of the Coefficient of Opacity. By E. A. Milne.

1. *Outline of the Paper.*—The recent work of Adams and Russell † brings forward evidence that the reversing layers of stars are not in thermodynamic equilibrium. This suggests a degree of caution in applying the fundamental method and formulæ of Saha to stellar spectra. Nevertheless, departure from thermodynamic equilibrium can only be found by pushing to as great a refinement as possible the theory which assumes thermodynamic equilibrium. At present the theory suffers from certain imperfections. The following work was largely completed before Adams and Russell's work appeared. Whether or no serious departures from thermodynamic equilibrium are ultimately found, it is hoped that the following treatment, which assumes strict thermodynamic equilibrium as far as ionization is concerned, may be of value.

* Cyclic quantum changes are not formally equivalent to monochromatic scattering, and if these occur, as Rosseland has pointed out (*Astrophys. Journ.*, **63**, 218, 1926), bright lines are possible.

† *Astrophys. Journ.*, **68**, 9, 1928.

Saha's explanation of the main features of stellar spectra in terms of high-temperature ionization involved the consideration of (*a*) the *concentration* of atoms of a given species in given states of ionization and excitation, (*b*) the *mean pressure* in the stellar atmosphere. The same considerations were involved in Fowler and Milne's quantitative treatment of the maxima of absorption lines. They determined the mean electron pressure by making the observed maxima agree with the calculated ones. But a rational treatment of stellar spectra should be capable of predicting the mean electron pressure from pure physics.

With regard to (*a*), the recent work of Unsöld, of Miss Payne, and of Adams and Russell determines the "*total number* of atoms above the photosphere," not the concentration. It is clear that this total number is what determines the form and intensity of an absorption line, and the preceding paper has attempted to give a precise meaning to the phrase in inverted commas. With regard to (*b*), the pressure, and hence the degree of ionization, must vary comparatively rapidly through the reversing layer. A rational theory must take account accurately of the varying ionization from level to level. Adams and Russell have stated the mean values they use in the form of integrals, but they have not evaluated them.

The method of the present paper provides in the first instance formulæ for the total number* $N_r^{(s)}$ of atoms in a stellar atmosphere, in any given state of ionization r and excitation s, down to any depth, in terms of the actual electron pressure P *at* that depth. The principal parameter is thus not the mean pressure through the layer, which is not required, but the electron-pressure at the base of the layer. Average pressures are nowhere used or needed.

It is necessary to take a mean value for the temperature T; but this probably varies within comparatively narrow limits. It is necessary further to take a constant value for gravity g. This is equivalent either to neglecting general and selective radiation-pressure (and so neglecting the chromospheric part of the reversing layer and its effect on line-intensity) or to assuming a mean value for effective gravity $\bar{g}$. The accurate calculation of selective radiation-pressure in the reversing layer involves the solution of complicated equations, and it seems best, in the first instance, to ignore it. But the pressure is far and away the most rapidly changing variable through the depth of a stellar atmosphere, and with it the degree of ionization must change rapidly. The formulæ of the present paper take accurate account of the change of ionization from level to level with pressure; they apply rigorously to an isothermal atmosphere of constant composition under gravitational equilibrium.

It is found that we are led to formulæ of the type

$$N_r^{(s)} = \frac{\epsilon}{mg} f(P, T) \quad . \quad . \quad . \quad . \quad (1)$$

* In the present paper the symbol r when used as a suffix to N denotes a state of ionization. In all other cases r denotes an intensity-ratio. No confusion is caused by the two usages.

where m is the atomic mass, ϵ is the proportion by mass of the element in question in the stellar atmosphere (absolute abundance), and f is an explicitly evaluated function which depends on the ionization and excitation potentials concerned. The function f depends on the circumstances of each problem—whether, for example, two or more stages of ionization are present, whether the majority of the atoms are just becoming ionized or are already once or more times ionized, whether the ionization potential of the atoms in question differs appreciably from the mean ionization potential of the atoms with which they are mixed, and so forth.

The ratio of any two expressions of the type (1), under any given circumstances, provides an expression of the type

$$\frac{N_{r+1}}{N_r} = \frac{f_{r+1}(P, T)}{f_r(P, T)} \quad . \quad . \quad . \quad . \quad (1a)$$

where N_r is the number of r-fold ionized atoms down to level P. Such formulæ may be termed "generalised Saha formulæ." Analytically they differ considerably from the corresponding simple Saha formulæ; they differ also in that P is the actual electron-pressure at the base of the layer, not a mean value. Formulæ of the type (1a) should always be used when deducing the total number N of atoms ($= N_0 + N_1 + \ldots$) from determinations of N_1, etc., by the method of Unsöld.

We are further led to formulæ for the optical depth τ down to this same level P. Two alternative sets of formulæ are found. In the first set the coefficient of general opacity κ is taken as constant; we denote it by $\bar{\kappa}$ to indicate that it is a mean for the layer concerned. In the second set, κ is taken as proportional to the electron-pressure at each level. I have shown in a previous paper* that if we adopt photo-electric ejection of electrons as the dominant cause of general opacity in the photospheric regions of a star, then it follows from a formula due to Kramers that, apart from a factor of the order of unity, $\kappa \propto P/T^{\frac{7}{2}}$. This is merely Eddington's formula $\kappa \propto \rho/T^{7/2}$ for the stellar interior, suitably modified to take account of the less complete ionization in the photospheric layers. We write the formula $\kappa = \alpha P/(kT)^{\frac{7}{2}}$, where for the moment P denotes the electron-pressure at *any* level. The two assumptions for κ lead to formulæ of the type

$$\tau = \frac{\bar{\kappa}}{g}\phi_1(P, T) \quad . \quad . \quad . \quad . \quad (2)$$

$$\tau = \frac{\alpha}{g}\phi_2(P, T) \quad . \quad . \quad . \quad . \quad (2')$$

where now P denotes as before the electron-pressure at the base of the layer, *i.e.* at the level τ. The functions ϕ_1 and ϕ_2 are explicitly evaluated. They depend on the ionization potentials concerned, and vary with the type of problem considered.

It will be sufficient first to consider (2). We now put $\tau = \tau_0$, the

* *M.N.*, **85**, 750, 1925, formulæ (3) and (24); 771, 1925, formula (2).

value found in the preceding paper as corresponding to the determination of $N_r^{(s)}$ by the line-contour method of Unsöld. Let P_0 be the value of P at the level $\tau = \tau_0$. Then we have

$$N_r^{(s)} = \frac{\epsilon}{mg} f(P_0, T) \quad . \quad . \quad . \quad . \quad (3)$$

$$\tau_0 = \frac{\bar{\kappa}}{g} \phi_1(P_0, T) \quad . \quad . \quad . \quad . \quad (4)$$

Calculations of ionization in stellar atmospheres are usually made by assuming a value for the mean electron-pressure, but in the present case we make no such assumption. P_0 remains to be evaluated. It is clear that if $\bar{\kappa}$ were known, (4) would determine P_0, and introducing it in (3) we could examine the behaviour of $N_r^{(s)}$ as a function of T. As it is we have two equations for the three unknowns ϵ, $\bar{\kappa}$, P_0, even if $N_r^{(s)}$ is fully determined by the method of Unsöld.

We shall now show that the consideration of the temperature in the stellar sequence at which a line attains its maximum intensity allows us to determine $\bar{\kappa}$ and P_0 without even a determination of $N_r^{(s)}$ by Unsöld's method. If, in addition, we know $N_r^{(s)}$ from a line-contour, (3) determines ϵ, the absolute abundance.

We assume for simplicity ϵ, $\bar{\kappa}$, and g to be constant along the stellar sequence.* Differentiate (3) for a maximum with regard to T, considering P_0 to be a function of T given by (4). The resulting equation is of the form

$$F_1(P_0, T_{max}) = 0 \quad . \quad . \quad . \quad . \quad (5)$$

the function F_1 involving neither ϵ, nor g, nor $\bar{\kappa}$. Solving it we find the value of P_0 at T_{max}. Inserting this value of P_0 in (4) and putting $T = T_{max}$ we have at once the value of $\bar{\kappa}$, determined purely from observation, namely, observation of T_{max}. With the same value of P_0, from (3) we can calculate the value of $N_r^{(s)}/\epsilon$ at T_{max}. Comparing it with a line-contour determination of $N_r^{(s)}$ we have at once the value of ϵ.

The value of $\bar{\kappa}$ thus determined can, however, now be used at other temperatures. For any T (4) can be solved for P_0, and the resulting value inserted in (3). We can then study the theoretical behaviour of $N_r^{(s)}$ at all temperatures, and by comparison with line-contour determinations obtain the value of ϵ for any element in any star.

Exactly the same procedure may be adopted if we use the theoretically preferable relation $\kappa = \alpha P/(kT)^{\frac{1}{2}}$ instead of κ = constant. We then have

$$N_r^{(s)} = \frac{\epsilon}{mg} f(P_0, T) \quad . \quad . \quad . \quad . \quad (6)$$

$$\tau_0 = \frac{\alpha}{g} \phi_2(P_0, T) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

* As regards g, we must if necessary observe the maximum for a sub-sequence of stars of constant g.

We now differentiate (6) with regard to T for a maximum, taking P_0 to be a function of T given by (7). The resulting equation is of the form

$$F_2(P_0, T_{max}) = 0 \quad . \quad . \quad . \quad . \quad (8)$$

the function F_2 involving neither ϵ, nor g, nor α. Solving it we obtain P_0 at $T_{max.}$ Inserting this value in (7) we obtain the value of the constant α directly from an observed $T_{max.}$ α should be a universal physical constant, and we can test its constancy by considering the maxima of different lines. We can also compare the value obtained astrophysically with the theoretical value derived from Kramers' formula. We thus have an astrophysical test of the Kramers theory.

Again, as before, once α is found we can solve (7) for P_0 at any temperature and calculate $N_r^{(s)}$ for any star from (6). We can then determine the value of ϵ from a line-contour.

It is of course not necessary to assume that g is constant along the stellar spectral sequence. If we confine attention to stars on the main sequence, g will be a function of temperature only, and this function could be inserted in our equations before differentiating with respect to T. We shall not, however, in the present paper introduce this refinement.

The earlier form of the theory of maxima of absorption lines in stellar spectra gave a value for $\bar{P}$, the mean pressure in the reversing layer at $T_{max.}$ The present formulæ for P_0—the roots of (5) or (8)—differ in general from the previously found formulæ for $\bar{P}$, but in most cases give a value of the same order of magnitude. In one case—called later Problem I.—the formulæ for P_0 and $\bar{P}$ are identical, but of course the meaning of P_0 is different from that of $\bar{P}$.

The function $\phi_1(P, T)$ has an interesting physical meaning apart from its occurrence in the expression for τ when κ is constant. Let N be the total number of atoms of the given species, in all states of ionization and excitation, down to the level where the electron-pressure is P. Let p be the total pressure at the level. Then we have

$$Nmg = \epsilon p \quad . \quad . \quad . \quad . \quad (9)$$

and further the well-known formula

$$\tau = \bar{\kappa} p/g \quad . \quad . \quad . \quad . \quad (10)$$

Hence

$$Nm/\tau = \epsilon/\bar{\kappa},$$

or by (2)

$$N = \frac{\epsilon}{mg}\phi_1(P, T) \quad . \quad . \quad . \quad . \quad (11)$$

Dividing by (1) we have

$$\frac{N_r^{(s)}}{N} = \frac{f(P, T)}{\phi_1(P, T)} \quad . \quad . \quad . \quad . \quad (12)$$

This is the complete generalisation of Saha's formula when we need total numbers of atoms instead of concentrations. The value of P

has to be determined from (4) or (7), once the constants $\bar{\kappa}$ or α have been found by the method of maxima.

We can next examine the effect of absolute magnitude for given spectral type,* *i.e.* the effect of varying g for given T. We now differentiate (3) and (4), or (6) and (7), with respect to g, keeping T constant and treating P as a function of g. Equations (3) and (4), for instance, give

$$\frac{d\mathrm{N}_r^{(s)}}{\mathrm{N}_r^{(s)}} = \frac{dg}{g}\left[-1 + \frac{\phi_1}{f}\frac{\partial f/\partial \mathrm{P}_0}{\partial \phi_1/\partial \mathrm{P}_0}\right] . \quad . \quad . \quad (13)$$

The sign of the expression in [ ] determines whether the line decreases or increases in intensity with absolute magnitude. If the coefficient of dg/g is positive, $\mathrm{N}_r^{(s)}$ increases as g increases, and the line is strengthened as we pass from giant to dwarf ; if negative, the line is weakened. We shall find that in a number of cases the sign of the coefficient of dg/g is different according as we adopt κ = const. or $\kappa \propto \mathrm{P}$. In particular it is found that if κ = const., the Balmer lines should be weakened as we pass from dwarfs to giants, but if $\kappa \propto \mathrm{P}$ they should be strengthened. Again, the theory predicts that if κ = const., the Sr^+ lines should be weakened as we pass from dwarf to giant, but if $\kappa \propto \mathrm{P}$ they should be strengthened. The known facts thus decide in favour of the law $\kappa \propto \mathrm{P}$ as against κ = constant.

Discussion of the details of the absolute magnitude effects is postponed to a later paper.

2. *Preliminary Formulæ.*—Let ρ be the density at any level, ν the number of atoms of the given element per cm.^{3} at that level, ϵ their proportion by mass, m their atomic mass. Then

$$m\nu = \epsilon\rho \quad . \quad . \quad . \quad . \quad (14)$$

Let h be the depth, measured inwards from a suitable reference level. The equation of mechanical equilibrium on the assumptions already made is

$$\frac{dp}{dh} = g\rho \, . \quad . \quad . \quad . \quad . \quad (15)$$

where p is the total pressure at h.

Let x_0, x_1, x_2 . . . be the fractions of neutral atoms, once-ionized atoms, . . . of the given element at any level. Let N be the total number of atoms of this element *down to* level h, per cm.^{2}, N_0 the number of neutral atoms *down* to h, N_1 that of once-ionized atoms, etc. Then we have

$$\mathrm{N} = \int \nu dh \quad . \quad . \quad . \quad . \quad (16)$$

$$\mathrm{N}_0 = \int \nu x_0 dh \quad . \quad . \quad . \quad . \quad (16')$$

$$\mathrm{N}_1 = \int \nu x_1 dh \quad . \quad . \quad . \quad . \quad (16'')$$

* Preferably, given colour-type.

and so on. Using (14) and (15) these become

$$N = \frac{\epsilon}{mg}\int_0^p dp = \frac{\epsilon p}{mg} \quad . \quad . \quad . \quad . \quad (17)$$

$$N_0 = \frac{\epsilon}{mg}\int_0^p x_0 dp \quad . \quad . \quad . \quad . \quad (17')$$

$$N_1 = \frac{\epsilon}{mg}\int_0^p x_1 dp \quad . \quad . \quad . \quad . \quad (17'')$$

In taking ϵ outside the sign of integration we have assumed it constant. It would, of course, be possible to conduct the investigation assuming the gases separate out, in virtue of their differing atomic weights, according to Dalton's law, but the reversing layer is probably too well mixed by convection currents and sun-spots for this to be possible. Selective radiation-pressure, which becomes of importance in the outer fringe of the reversing layer, is of course another matter. It seems best to take a constant mean value for ϵ as a first approximation.

Let P be the electron-pressure at level h. The typical ionization formula may be written

$$\frac{x_1}{x_0}P = K_1 \quad . \quad . \quad . \quad . \quad (18)$$

where K_1 is a function of the temperature T given by

$$K_1 = \frac{q_1}{q_0}\frac{(2\pi m_e)^{\frac{3}{2}}}{h^3}(kT)^{\frac{5}{2}}e^{-\chi_1/kT} \quad . \quad . \quad . \quad (19)$$

Here m_e is the mass of the electron, χ_1 is the first ionization potential, q_0, q_1 . . . are the statistical weights of the neutral, once-ionized . . . atoms in their normal states. We have ignored a very slowly varying function of T approximately equal to unity.

Our general method is to express p, x_0, x_1 . . . in terms of P and then to evaluate the integrals (17′), (17″) . . ., taking T and hence K_1, K_2 . . . as constant. The expressions for p, x_0, x_1 . . . are different according to circumstances. We have classified these circumstances into five cases, each producing its own problem.

The great advantage of choosing P as the main independent variable is that all the integrals come out as rational or logarithmic functions of P. All surds are avoided, and the subsequent calculations are unexpectedly simple.

Calculation of N, N_0, N_1 . . .

3. *Problem I.—Only neutral and once-ionized atoms present. Material with a mean ionization potential equal to that of the particular atoms considered.* This may be regarded as the basic problem. It is identical with the problem of a stellar atmosphere composed of only one kind of atom. It applies to stars of low effective temperature in which ionization is just commencing.

We write here

$$x_1 = x, \qquad x_0 = 1 - x,$$

and we have

$$\frac{P}{p} = \frac{x}{1+x} \quad . \quad . \quad . \quad . \quad (20)$$

$$\frac{x}{1-x}P = K_1 \quad . \quad . \quad . \quad . \quad (21)$$

Solving (20) and (21) we have

$$x = \frac{K_1}{P+K_1}, \qquad p = \frac{P(P+2K_1)}{K_1} \quad . \quad . \quad . \quad (22)$$

Hence

$$N = \frac{\epsilon}{mg}\frac{P(P+2K_1)}{K_1} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$N_0 = \frac{\epsilon}{mg}\int_0^P \frac{P}{P+K_1} d\left(\frac{P(P+2K_1)}{K_1}\right) = \frac{\epsilon}{mg}\frac{P^2}{K_1} \quad . \quad (23')$$

$$N_1 = \frac{\epsilon}{mg}\int \frac{K_1}{P+K_1} d\left(\frac{P(P+2K_1)}{K_1}\right) = \frac{\epsilon}{mg}2P \quad . \quad . \quad (23'')$$

From (23′) and (23″) we have the generalised Saha formula

$$\frac{N_1}{N_0} = \frac{K_1}{\frac{1}{2}P} \quad . \quad . \quad . \quad . \quad (24)$$

The simple Saha formula for the concentrations ν_0, ν_1 is

$$\frac{\nu_1}{\nu_0} = \frac{K_1}{\bar{P}},$$

$\bar{P}$ being the mean electron-pressure. Hence in Problem I. the formula for *concentrations* gives the correct result for *totals* if we take $\bar{P} = \frac{1}{2}P$ and ignore varying ionization from level to level. To this circumstance appears to be due the success of the crude Saha formula in its application to stellar atmospheres.

4. *Problem II.—All the atoms once-ionized. Second stage setting in. Mean second-stage ionization potential equal to that of atoms in question.* This is the problem of the atmosphere composed of a single element at somewhat higher temperatures than in Problem I.

We write here

$$x_0 = 0, \qquad x_1 = 1-y, \qquad x_2 = y \quad . \quad . \quad (25)$$

$$\frac{P}{p} = \frac{1+y}{2+y} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$\frac{y}{1-y}P = K_2 \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Hence

$$y = \frac{K_2}{P+K_2} \quad . \quad . \quad . \quad . \quad (28)$$

$$p = P\frac{2P+3K_2}{P+2K_2} \quad . \quad . \quad . \quad . \quad (29)$$

We find then

$$N = \frac{\epsilon}{mg}\frac{P(2P + 3K_2)}{P + 2K_2} \quad . \quad . \quad . \quad (30)$$

$$I_1 = \frac{\epsilon}{mg}\int_0^P \frac{P}{P + K_2}\,d\left(\frac{P(2P + 3K_2)}{P + 2K_2}\right) = \frac{2\epsilon}{mg}\left[\frac{P(K_2 + P)}{2K_2 + P} - K_2 \log\left(1 + \frac{1}{2}\frac{K_2}{P}\right)\right] \quad (30')$$

$$I_2 = \frac{\epsilon}{mg}\int_0^P \frac{K_2}{P + K_2}\,d\left(\frac{P(2P + 3K_2)}{P + 2K_2}\right) = \frac{2\epsilon}{mg}\left[\frac{\frac{1}{2}PK_2}{2K_2 + P} + K_2 \log\left(1 + \frac{1}{2}\frac{P}{K_2}\right)\right] \quad (30'')$$

The generalised Saha formula is

$$\frac{N_2}{N_1} = K_2\frac{\frac{1}{2}P + (2K_2 + P)\log(1 + \frac{1}{2}P/K_2)}{P(K_2 + P) - K_2(2K_2 + P)\log(1 + \frac{1}{2}P/K_2)} \quad . \quad (31)$$

Comparing this with the crude Saha formula

$$\frac{\nu_2}{\nu_1} = \frac{K_2}{\bar{P}}$$

we have

$$\bar{P} = \frac{P(K_2 + P) - K_2(2K_2 + P)\log(1 + \frac{1}{2}P/K_2)}{\frac{1}{2}P + (2K_2 + P)\log(1 + \frac{1}{2}P/K_2)} \quad . \quad (32)$$

For $P = K_2$, this gives $\bar{P} = 0{\cdot}46P$. We find also from (32) that for P small, $\bar{P} \sim \frac{1}{2}P$; for P large, $\bar{P} \sim P/\log(\frac{1}{2}P/K_2)$.

5. *Problem IIa.—All the atoms* $(n-1)$*-times ionized. nth stage setting in. Mean nth I.P. the same as for the particular atoms considered.* The formulæ are found to be

$$p = P\frac{nP + (n+1)K_n}{(n-1)P + nK_n},$$

$$N = \frac{\epsilon}{mg}\frac{P(nP + (n+1)K_n)}{(n-1)P + nK_n} \quad . \quad . \quad . \quad (33)$$

$$N_{n-1} = \frac{n\epsilon}{mg}\left[\frac{P(K_n + P)}{nK_n + (n-1)P} - \frac{K_n}{n-1}\log\left(1 + \frac{n-1}{n}\frac{P}{K_n}\right)\right] \quad . \quad (33')$$

$$N_n = \frac{n\epsilon}{mg}\left[\frac{1}{n}\cdot\frac{PK_n}{nK_n + (1-1)P} + \frac{K_n}{n-1}\log\left(1 + \frac{n-1}{n}\frac{P}{K_n}\right)\right] \quad (33'')$$

These are simply given for completeness.

6. *Problem III.—The atoms in question just becoming ionized in the presence of an excess of atoms already once ionized.* This is the problem of an atom with an ionization potential considerably greater than those of the atoms with which it is surrounded. It corresponds to hydrogen in A-type stars.

We write here

$$x_1 = x, \qquad x_0 = 1 - x$$

and we have

$$P = \tfrac{1}{2}p \quad . \quad . \quad . \quad . \quad . \quad (34)$$

since the free electrons form approximately one-half of the total number of particles.

We find

$$N = \frac{\epsilon}{mg} 2P \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

$$N_0 = \frac{2\epsilon}{mg}\left[P - K_1 \log\left(1 + \frac{P}{K_1}\right)\right]. \quad . \quad . \quad (35')$$

$$N_1 = \frac{2\epsilon}{mg} K_1 \log\left(1 + \frac{P}{K_1}\right) \quad . \quad . \quad . \quad . \quad (35'')$$

Here

$$\frac{N_1}{N_0} = \frac{K_1 \log (1 + P/K_1)}{P - K_1 \log (1 + P/K_1)} \quad . \quad . \quad . \quad (36)$$

whence by comparison with the crude Saha formula we find

$$\bar{P} = \frac{P - K_1 \log (1 + P/K_1)}{\log (1 + P/K_1)} \quad . \quad . \quad . \quad (37)$$

When $P = K_1$, this gives $\bar{P} = 0{\cdot}44P$. When P is small, $\bar{P} \sim \frac{1}{2}P$; when P is large, $\bar{P} \sim P/\log (1 + P/K_1)$.

7. *Problem IIIa.—The atoms in question just becoming ionized in the presence of an excess of atoms already* n*-times ionized.*

We have here

$$P = \frac{n}{n+1} p \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

$$N = \frac{\epsilon}{mg} \frac{n+1}{n} P \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

$$N_0 = \frac{n+1}{n} \frac{\epsilon}{mg}\left[P - K_1 \log\left(1 + \frac{P}{K_1}\right)\right] \quad . \quad . \quad (39')$$

$$N_1 = \frac{n+1}{n} \frac{\epsilon}{mg} K_1 \log\left(1 + \frac{P}{K_1}\right) \quad . \quad . \quad . \quad (39'')$$

The formulæ of §§ 6, 7 apply to any stage of ionization of the atoms in question, on replacing K_1 by K_2, ...

8. *Problem IV.—The atoms in question becoming ionized with ionization potential* χ_1 *in the presence of an excess of atoms becoming ionized with ionization potential* $\bar{\chi}_1$. This is the fundamental practical problem of stellar atmospheres at not too large temperatures, when in effect second-stage ionization has not commenced.

Let $\bar{K}_1$ be the ionization constant calculated with $\bar{\chi}_1$. Let $\bar{x}$ be the degree of ionization of the majority of the atoms. Then

$$\frac{x}{1-x} P = K_1, \qquad \frac{\bar{x}}{1-\bar{x}} P = \bar{K}_1 \quad . \quad . \quad (40)$$

$$\frac{P}{p} = \frac{\bar{x}}{1+\bar{x}} \quad . \quad . \quad . \quad . \quad (41)$$

whence

$$p = \frac{P(P + 2\bar{K}_1)}{\bar{K}_1} \quad . \quad . \quad . \quad . \quad (42)$$

We find

$$N = \frac{\epsilon}{mg} \frac{P(P + 2\bar{K}_1)}{\bar{K}_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

$$N_0 = \frac{2\epsilon}{mg} \frac{1}{\bar{K}_1}\left[\tfrac{1}{2}P^2 + (\bar{K}_1 - K_1)\left\{P - K_1 \log\left(1 + \frac{P}{\bar{K}_1}\right)\right\}\right] \quad (43')$$

$$N_1 = \frac{2\epsilon}{mg} \frac{K_1}{\bar{K}_1}\left[P + (\bar{K}_1 - K_1) \log\left(1 + \frac{P}{\bar{K}_1}\right)\right] \quad . \quad . \quad (43'')$$

9. *Problem V.—Three successive stages of ionization present. Material with mean first- and second-stage I.P.'s equal to those of the particular atoms considered.* This is the problem of the behaviour of the principal lines of an ionized atom. It is the next step in the problem of the stellar atmosphere composed of only one kind of atom.

We write here

$$x_1 = x, \qquad x_2 = y, \qquad x_0 = 1 - x - y,$$

and we have

$$\frac{x}{1 - x - y} P = K_1, \qquad \frac{y}{x} P = K_2 \quad . \quad . \quad (44)$$

Solving these we have

$$x = \frac{1}{1 + (P/K_1) + (K_2/P)}, \qquad y = \frac{K_2/P}{1 + (P/K_1) + (K_2/P)} . \quad (45)$$

We have also

$$\frac{P}{p} = \frac{x + 2y}{1 + x + 2y},$$

giving

$$p = P^2 \frac{1 + (P/K_1) + 3(K_2/P)}{1 + 2(K_2/P)} \quad . \quad . \quad . \quad (46)$$

Hence

$$N = \frac{\epsilon}{mg} P \frac{3K_2 + 2P + (P^2/K_1)}{2K_2 + P} \quad . \quad . \quad (47)$$

$$N_0 = \frac{\epsilon}{mg}\int_0^p (1 - x - y)dp = \frac{\epsilon}{mg}\int_0^P \frac{P/K_1}{1 + (P/K_1) + (K_2/P)} d\left(P \cdot \frac{3K_2 + 2P + (P^2/K_1)}{2K_2 + P}\right) \quad (47')$$

$$N_1 = \frac{\epsilon}{mg}\int_0^p x dp = \frac{\epsilon}{mg}\int_0^P \frac{1}{1 + (P/K_1) + (K_2/P)} d\left(P \cdot \frac{3K_2 + 2P + P^2/K_1}{2K_2 + P}\right) \quad (47'')$$

$$N_2 = \frac{\epsilon}{mg}\int_0^p y dp = \frac{\epsilon}{mg}\int_0^P \frac{K_2/P}{1 + (P/K_1) + (K_2/P)} d\left(P \cdot \frac{3K_2 + 2P + P^2/K_1}{2K_2 + P}\right) \quad (47''')$$

It is convenient to make the substitution

$$P = \theta K_1^{\frac{1}{2}} K_2^{\frac{1}{2}}$$

and to write

$$\beta = (K_1/K_2)^{\frac{1}{2}}.$$

The last three integrals then reduce to

$$N_0 = \frac{\epsilon(K_1K_2)^{\frac{1}{2}}}{mg}\int_0^\theta \frac{\theta^2}{1+\beta\theta+\theta^2} d\left(\theta\frac{\theta^3+2\beta\theta+\theta^2}{2+\beta\theta}\right) \quad . \quad (48')$$

$$N_1 = \frac{\epsilon(K_1K_2)^{\frac{1}{2}}}{mg}\int_0^\theta \frac{\beta\theta}{1+\beta\theta+\theta^2} d\left(\theta\frac{\theta^3+2\beta\theta+\theta^2}{2+\beta\theta}\right) \quad . \quad (48'')$$

$$N_2 = \frac{\epsilon(K_1K_2)^{\frac{1}{2}}}{mg}\int_0^\theta \frac{1}{1+\beta\theta+\theta^2} d\left(\theta\frac{\theta^3+2\beta\theta+\theta^2}{2+\beta\theta}\right) \quad . \quad (48''')$$

Now

$$d\left(\theta\frac{\theta^3+2\beta\theta+\theta^2}{2+\beta\theta}\right) = 2(1+\beta\theta+\theta^2)\left[\frac{1}{2+\beta\theta}+\frac{1}{(2+\beta\theta)^2}\right]d\theta.$$

Inserting this in the integrals (48), evaluating them and returning to the variable P, we find

$$N_0 = \frac{\epsilon}{mg}\frac{P^3}{K_1(P+2K_2)} \quad . \quad . \quad . \quad . \quad . \quad (49')$$

$$N_1 = \frac{2\epsilon}{mg}\left[\frac{P(P+K_2)}{P+2K_2} - K_2\log\left(1+\frac{1}{2}\frac{P}{K_2}\right)\right] \quad . \quad (49'')$$

$$N_2 = \frac{2\epsilon}{mg}\left[\frac{\frac{1}{2}PK_2}{P+2K_2} + K_2\log\left(1+\frac{1}{2}\frac{P}{K_2}\right)\right] \quad . \quad . \quad (49''')$$

As a check, we have

$$\begin{aligned} N_0+N_1+N_2 &= \frac{\epsilon}{mg}\left[\frac{P^3/K_1+2P(P+K_2)+PK_2}{P+2K_2}\right] \\ &= \frac{\epsilon}{mg}P\frac{P^2/K_1+2P+3K_2}{P+2K_2} \\ &= N, \end{aligned}$$

by (47), as it should be.

I cannot refrain from observing that (49″) and (49‴) are remarkable formulæ, for they show that the numbers of once-ionized and twice-ionized atoms down to level P are independent of K_1, *i.e.* are independent of the first I.P. χ_1. If we increase χ_1, the number of once-ionized atoms at any point is decreased, but P at that point is also decreased; formula (49″) shows that the compensation is perfect, and that the number of once-ionized atoms down to a given P is unaltered. It should be noted that (49″) and (49‴) are identical with formulæ (30′) and (30″) of Problem II., which is the particular case when $\chi_1 = 0$. Again, (49′) and (49″) reduce to (23′) and (23″) on putting $K_2 = 0$, as they should.

The generalised Saha formulæ are

$$\frac{N_1}{N_0} = K_1 \frac{P(P + K_2) - K_2(P + 2K_2)\log(1 + \frac{1}{2}P/K_2)}{\frac{1}{2}P^3} \quad . \quad (50')$$

$$\frac{N_2}{N_1} = K_2 \frac{\frac{1}{2}P + (P + 2K_2)\log(1 + \frac{1}{2}P/K_2)}{P(P + K_2) - K_2(P + 2K_2)\log(1 + \frac{1}{2}P/K_2)} \quad . \quad (50'')$$

Comparing these with the crude Saha formulæ

$$\frac{\nu_1}{\nu_0} = \frac{K_1}{\bar{P}}, \qquad \frac{\nu_2}{\nu_1} = \frac{K_2}{\bar{P}}$$

we have in the first case

$$\bar{P} = \frac{\frac{1}{2}P^3}{P(P + K_2) - K_2(P + 2K_2)\log(1 + \frac{1}{2}P/K_2)},$$

and in the second case

$$\bar{P} = \frac{P(P + K_2) - K_2(P + 2K_2)\log(1 + \frac{1}{2}P/K_2)}{\frac{1}{2}P + (P + 2K_2)\log(1 + \frac{1}{2}P/K_2)}.$$

The first case gives for P large, $\bar{P} \sim \frac{1}{2}P$; for P small, $\bar{P} \sim \frac{2}{3}P$. The second case gives for P large, $\bar{P} \sim P/\log(\frac{1}{2}P/K_2)$; for P small, $\bar{P} \sim \frac{1}{2}P$.

10. *Problem Va.—Three stages of ionization of the particular atoms considered, in the presence of an excess of atoms already once ionized.* Equation (46) is now replaced by

$$p = 2P,$$

and consequently we have

$$N = \frac{2\epsilon}{mg}P \quad . \quad . \quad . \quad . \quad (51)$$

$$N_0 = \frac{2\epsilon}{mg}\int_0^P \frac{P/K_1}{1 + (P/K_1) + (K_2/P)}dP = \frac{2\epsilon(K_1K_2)^{\frac{1}{2}}}{mg}\int_0^\theta \frac{\theta^2}{1 + \beta\theta + \theta^2}d\theta \quad (51')$$

$$N_1 = \frac{2\epsilon}{mg}\int_0^P \frac{1}{1 + (P/K_1) + (K_2/P)}dP = \frac{2\epsilon(K_1K_2)^{\frac{1}{2}}}{mg}\int_0^\theta \frac{\beta\theta}{1 + \beta\theta + \theta^2}d\theta \quad (51'')$$

$$N_2 = \frac{2\epsilon}{mg}\int_0^P \frac{K_2/P}{1 + (P/K_1) + (K_2/P)}dP = \frac{2\epsilon(K_1K_2)^{\frac{1}{2}}}{mg}\int_0^\theta \frac{1}{1 + \beta\theta + \theta^2}d\theta \quad (51''')$$

These integrals can easily be evaluated exactly, but since $\beta = (K_1/K_2)^{\frac{1}{2}}$ is always large (since $\chi_2 > \chi_1$) it is convenient to obtain approximations valid for β large. Returning to the variable P we find then

$$N_0 = \frac{2\epsilon}{mg} \cdot \frac{1}{2}\frac{P^2}{K_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (52')$$

$$N_1 = \frac{2\epsilon}{mg}\left[P - K_2\log\left(1 + \frac{P}{K_2}\right) - \frac{1}{2}\frac{P^2}{K_1}\right] \quad . \quad . \quad (52'')$$

$$N_2 = \frac{2\epsilon}{mg}K_2\log\left(1 + \frac{P}{K_2}\right) \quad . \quad . \quad . \quad . \quad (52''')$$

These hold on the assumption that $P/(K_1K_2)^{\frac{1}{2}}$ is not itself large.

11. The above are the principal problems requiring consideration. Others may be worked out similarly as they arise, but the above seem to cover the cases wanted.

Calculation of the Optical Depth.

12. (a) *Case of* κ = *constant* = $\bar{\kappa}$.—We have

$$\tau = \int_0^h \bar{\kappa}\rho dh = \frac{\bar{\kappa}}{g}p \quad . \quad . \quad . \quad . \quad (53)$$

The formulæ we have previously given for p in each problem thus suffice to determine τ without further calculation. We have also $N = \epsilon\tau/m\bar{\kappa}$, so that τ can equally well be found from the formulæ for N.

12. (b) *Case of* $\kappa = \alpha P/(kT)^{\frac{1}{2}}$.—We have

$$\tau = \int_0^h \kappa\rho dh = \frac{1}{g}\int_0^p \kappa dp.$$

As usual we adopt a constant mean value for T. It is convenient to put

$$\alpha/(k\mathrm{T})^{\frac{1}{2}} = \alpha'.$$

Then

$$\tau = \frac{\alpha'}{g}\int_0^p \mathrm{P}dp \quad . \quad . \quad . \quad . \quad (54)$$

All we have to do is to insert, for each problem, the expression for p in terms of P and carry out the integration. The results are :

Problem I.

$$\tau = \frac{\alpha'}{g}\mathrm{P}^2\left(1 + \frac{2}{3}\frac{\mathrm{P}}{\mathrm{K}_1}\right). \quad . \quad . \quad . \quad (55)$$

Problem II.

$$\tau = \frac{2\alpha'}{g}\left[\tfrac{1}{2}\mathrm{P}^2 - \mathrm{K}_2{}^2\log\left(1 + \frac{1}{2}\frac{\mathrm{P}}{\mathrm{K}_2}\right) + \frac{\mathrm{P}\mathrm{K}_2{}^2}{\mathrm{P} + 2\mathrm{K}_2}\right] \quad . \quad (55')$$

Problem IIa.

$$\tau = \frac{\alpha'}{g}\frac{n}{(n-1)^3}\left[\frac{(n-1)^2\mathrm{P}^2}{2} - \mathrm{K}_n{}^2\log\left(1 + \frac{n-1}{n}\frac{\mathrm{P}}{\mathrm{K}_n}\right) + \frac{(n-1)\mathrm{P}\mathrm{K}_n{}^2}{(n-1)\mathrm{P} + n\mathrm{K}_n}\right] \quad (55'')$$

Problem III.

$$\tau = \frac{\alpha'}{g}\mathrm{P}^2 \quad . \quad . \quad . \quad . \quad (55''')$$

Problem IIIa.

$$\tau = \frac{\alpha'}{g}\frac{n+1}{2n}\mathrm{P}^2 \quad . \quad . \quad . \quad . \quad (55^{\mathrm{iv}})$$

Problem IV.

$$\tau = \frac{\alpha'}{g}\mathrm{P}^2\left(1 + \frac{2}{3}\frac{\mathrm{P}}{\mathrm{K}_1}\right) \quad . \quad . \quad . \quad (55^{\mathrm{v}})$$

Problem V.

$$\tau = \frac{\alpha'}{g}\int_0^P P d\left(P\frac{3K_2 + 2P + (P^2/K_1)}{2K_2 + P}\right)$$

$$= \frac{2\alpha'}{g}K_1K_2\int_0^\theta \theta(1 + \beta\theta + \theta^2)\left(\frac{1}{2 + \beta\theta} + \frac{1}{(2 + \beta\theta)^2}\right)d\theta$$

$$= \frac{2\alpha'}{g}K_1K_2\left[\frac{\frac{1}{2}\theta^3}{\beta} + \frac{\beta^2 - 1}{2\beta^2}\theta^2 + \frac{4 - \beta^2}{\beta^4}\log(1 + \tfrac{1}{2}\beta\theta) - \frac{4 - \beta^2}{\beta^3}\frac{\theta}{2 + \beta\theta}\right].$$

The last expression is too complicated to be of much use as it stands. When θ is not large but β is large, it reduces to

$$\tau = \frac{\alpha'}{g}P^2\left(1 + \frac{2}{3}\frac{P}{K_1}\right) \quad . \quad . \quad . \quad (55^{vi})$$

Problem Va.

$$\tau = \frac{\alpha'}{g}P^2 \quad . \quad . \quad . \quad . \quad (55^{vii})$$

Maximum Problems.

13. *The Number of Excited Atoms.*—If $N_0^{(s)}$ is the number of neutral excited atoms, of statistical weight $q_0^{(s)}$, in the state $\chi_1^{(s)}$, then

$$\frac{N_0^{(s)}}{N_0} = \frac{q_0^{(s)}}{q_0}e^{-(\chi_1 - \chi_1^{(s)})/kT} \quad . \quad . \quad . \quad . \quad (56)$$

We shall denote the expression on the right-hand side by $A_0^{(s)}$. We have similar expressions for excited ionized atoms.

We now put $\tau = \tau_0$, $P = P_0$ and determine for what temperature $N_0^{(s)}$ is a maximum subject to $\tau_0 =$ constant, P_0 being a function of T given by $\tau_0 =$ constant. We shall work out the results first for $\kappa =$ constant $= \bar{\kappa}$, afterwards for $\kappa = \alpha' P$. The working will be given in detail for Problem I. and an outline given for the other problems.

14. *Problem I., with* $\kappa =$ *constant* $= \bar{\kappa}$.—By (23′) we have

$$N_0^{(s)} = \frac{\epsilon}{mg}A_0^{(s)}\frac{P_0^2}{K_1} \quad . \quad . \quad . \quad (57)$$

and by (22) and (53),

$$\tau_0 = \frac{\bar{\kappa}}{g}P_0\left(2 + \frac{P_0}{K_1}\right) \quad . \quad . \quad . \quad (58)$$

Differentiating (57) logarithmically with regard to T we have

$$\frac{dN_0^{(s)}}{N_0^{(s)}} = \frac{dA_0^{(s)}}{A_0^{(s)}} + \frac{2dP_0}{P_0} - \frac{dK_1}{K_1} \quad . \quad . \quad . \quad (59)$$

and similarly from (58),

$$0 = dP_0\left(2 + 2\frac{P_0}{K_1}\right) - \frac{P_0^2}{K_1^2}dK_1 \quad . \quad . \quad (60)$$

Eliminating dP_0 between (59) and (60) we find that $N_0^{(s)}$ will be a maximum, where

$$\frac{1}{1 + P_0/K_1}\frac{dK_1}{K_1} - \frac{dA_0^{(s)}}{A_0^{(s)}} = 0 \quad . \quad . \quad . \quad (61)$$

Now

$$\frac{dK_1}{K_1} = \frac{\chi_1 + \frac{5}{2}kT}{kT^2}dT \quad . \quad . \quad . \quad (62)$$

$$\frac{dA_0^{(s)}}{A_0^{(s)}} = \frac{\chi_1 - \chi_1^{(s)}}{kT^2}dT \quad . \quad . \quad . \quad (63)$$

Inserting these in (61) and solving for P_0 we have

$$P_0 = \left[K_1\frac{\chi_1^{(s)} + \frac{5}{2}kT}{\chi_1 - \chi_1^{(s)}}\right]_{T=T_{max}} \quad . \quad . \quad . \quad (64)$$

This gives the electron-pressure at depth τ_0 when the absorption line attains its maximum intensity. It is one of the principal results of the present paper.

The "maximum formula" found by the method of Fowler and Milne * was

$$\bar{P}_e = \left[K_1\frac{\chi_1^{(s)} + \frac{5}{2}kT}{\chi_1 - \chi_1^{(s)}}\right]_{T=T_{max}} \quad . \quad . \quad . \quad (65)$$

We see that it is identical with (64) save that P_0 now replaces $\bar{P}_e$, the "mean pressure of electrons in the reversing layer." We shall find that it is only in this Problem I. (and only here when κ = constant) that the two methods give similar formulæ. But the significance of the formulæ is quite different in the two methods. In the earlier method $\bar{P}_e$ was taken as constant. In the present method P_0 is a function of T given by (58), and, moreover, P_0 is not a mean but the electron-pressure at a definite depth, namely, the level $\tau = \tau_0$. The present method is thus of much greater precision in its result. Further, the new formula is of much greater precision in its mode of derivation, since changes of ionization from level to level have been accurately allowed for.

Using this value of P_0 in (58) we have the value of $\bar{\kappa}$,

$$\bar{\kappa} = \left[\frac{\tau_0 g}{K_1}\frac{(\chi_1 - \chi_1^{(s)})^2}{(\chi_1^{(s)} + \frac{5}{2}kT)(2\chi_1 - \chi_1^{(s)} + \frac{5}{2}kT)}\right]_{T=T_{max}} \quad . \quad (66)$$

This affords the numerical value of $\bar{\kappa}$ by means of an observed T_{max}. If $\bar{\kappa}$ were accurately constant for a sequence of stars of given g, and if its value were known from pure physics, equation (66) would determine the value of T_{max} under the conditions of Problem I. for lines of I.P. χ_1 and E.P. $\chi_1 - \chi_1^{(s)}$. T_{max} would thus be predictable from pure physics without any empirical assumption as to pressures in stellar atmospheres.

* *M.N.*, **83**, 403, 1923. The formula is readily deduced by making $A_0^{(s)}(1 - \bar{x})$ or $A_0^{(s)}\bar{P}_e/(\bar{P}_e + K_1)$ a maximum subject to $\bar{P}_e$ being constant. Differentiation gives $dA_0^{(s)}/A_0^{(s)} - dK_1/(\bar{P}_e + K_1) = 0$, which is of the same form as (61).

Again, insertion of the value of P_0 in (57) determines the value of $N_0^{(s)}/\epsilon$ at T_{max}. The determination of $N_0^{(s)}$ at T_{max} by the line-contour method of Unsöld would then determine ϵ, the absolute abundance of the element by mass in the stellar atmosphere.

15. *The Definition of "Maximum."*—Before we discuss numerical values, there are one or two points to discuss which require rather refined considerations. They occur with each type of problem, and they occur whether we take κ = const. = $\bar{\kappa}$ or $\kappa = \alpha P/(kT)^{9/2}$. It is therefore best to dispose of them now.

There is nothing to prevent us from considering a hypothetical series of stars all having rigorously the same value of $\bar{\kappa}$. Now (66) involves τ_0, but the numerical value of $\bar{\kappa}$ cannot depend on τ_0. It follows that the value of T_{max} (which is the root of (66)) must depend on the value of τ_0. Hence the value of P_0 at T_{max}, given by (64), must depend on τ_0, although τ_0 does not explicitly appear in (64). Likewise the value of $N_0^{(s)}$ at T_{max} must depend on τ_0.

But the value of τ_0 is to some extent arbitrary. We have seen in the preceding paper that if we fix attention on the part of the line-contour where r, the intensity-ratio, is $\frac{1}{2}$, the value of τ_0 is approximately $\frac{1}{3}$* ; as we move out towards the wings τ_0 tends to approximately $\frac{1}{2}$* ; as we approach the centre, τ_0 decreases from $\frac{1}{3}$ downwards, taking smaller and smaller values the blacker is the line-centre. The important point is that τ_0 is a function of r and is constant only when r remains constant.

We are now in a position to define exactly what we mean by the "maximum" of an absorption line. Fix attention on that point Q′ of the line-contour where the intensity-ratio r has some assigned value r'. Let ν' be the frequency at this point, ν_0 the frequency at the line-centre Q_0. As the line increases in intensity, Q′ separates more and more from Q_0 ; it eventually reaches its maximum separation from Q_0 and then approaches Q_0 again. To r' corresponds a definite optical depth τ_0' measured in general opacity. The frequency ν' will therefore be at its maximum separation from ν_0 when $N_0^{(s)}(\tau_0')$, the number of atoms down to τ_0', attains its maximum. We have seen that the temperature at which $N_0^{(s)}(\tau_0')$ attains its maximum depends on τ_0'. Thus the temperature at which Q′ attains its maximum separation from Q_0 depends on τ_0', and is accordingly different for different parts of the line-contour. This is because different parts of the line-contour correspond to different depths, *i.e.* to different pressures, and the ionization varies with pressure.

The argument can be made more precise and perhaps more immediate as follows : If a_ν is the *atomic* scattering coefficient at frequency ν, then by Schuster's formula, for a fixed value of r' the value of the product $N_0^{(s)}a_\nu$ is constant. Now a_ν decreases as ν separates from ν_0. Hence as T increases and accordingly $N_0^{(s)}$ increases, a_ν must decrease and so ν separates from ν_0. At T_{max}, $N_0^{(s)}$ is a maximum for constant r', a_ν is a minimum (for constant r'), and $|\nu - \nu_0|$ is a maximum. Thus

* These values vary slightly with frequency and temperature, as shown in the preceding paper.

T_{max} has the property that for given intensity-ratio r' the line is widest at T_{max}.

Thus the phrase "maximum of an absorption-line" has by itself no precise meaning. It becomes precise when we fix attention on a point Q′ of the line-contour where the intensity-ratio has some assigned value r'. We then mean by the temperature (or spectral class) at which the "line" attains its "maximum" that temperature (or spectral class) at which Q′ has its maximum separation in frequency from the line-centre Q_0. The value of T_{max} depends on the chosen value r'. The degree of variation of T_{max} with τ_0' or with r' is of course slight, but it is readily found from (66) that T_{max} increases as τ_0 increases, *i.e.* as r' increases, *i.e.* as we move out towards the wings of the line. The further out towards the wings the point on which we fix attention, the greater is the pressure to which we see down into the star and the greater is T_{max}.

The precise value of τ_0' for given r' as calculated in the preceding paper depends on certain special assumptions, but the considerations of the last few paragraphs are perfectly general. They do not depend on any special assumptions as to the structure of the stellar atmosphere or the behaviour of the general and selective coefficients. They are quite independent of the validity of the numerical values of constants required in Unsöld's method. They apply whatever the law of variation of the atomic absorption (scattering) coefficient in the neighbourhood of ν_0. All we have to do is to measure for each star the separation $\nu' - \nu_0$ of the point Q′ (where the intensity-ratio has some assigned value r') from Q_0 and to ascertain the temperature or spectral class at which $(\nu' - \nu_0)$ attains its maximum value.

As to the numerical value of τ_0', it can be calculated from the quadratic equation given in the preceding paper on the assumptions there made. This is the value of τ_0 to use in equations (58) and (66). It is only in the calculation of τ_0' for given r' that special assumptions are introduced. The value found for τ_0' probably depends very little on these assumptions, and we may be content for the time being with the value found in the preceding paper.

If we are working on the basis $\kappa = \alpha P/(kT)^{9/2}$ we simply use the same value of τ_0' in a different but corresponding set of equations. The precise calculation of τ_0 for given r taking accurate account of the variation with depth of κ the general absorption coefficient and s_ν the selective mass-scattering coefficient of the material as a whole is only a matter of mathematics; we have to solve the differential equation (3) of the preceding paper under more general conditions. I hope to deal with these refinements on a future occasion.

If the arguments of the preceding paragraphs are accepted, the existing loose determinations of "maxima" of absorption lines become obsolete. A great deal of observational work is required for the re-determination of maxima, in the new precise sense, from line-contours. Until this is done, it does not seem worth while trying to make an exhaustive test of the formulæ with the existing data. In any case the present paper, dealing as it does with the principles

involved, is scarcely the place for detailed numerical work; also, others have better access to reliable data. It is worth while, however, just to apply our formulæ to two stars, the Sun and Capella.

16. *Numerical Values.*—The Sun and Capella are of the same type, Go. Their effective temperatures are

$$\text{Sun, } 5740^\circ, \qquad \text{Capella, } 5200^\circ,$$

and the values of g are

$$\text{Sun, } 2{\cdot}74 \times 10^4 \text{ cm. sec.}^{-2}, \qquad \text{Capella, } 6{\cdot}08 \times 10^2 \text{ cm. sec.}^{-2}.$$

We will take it from observation that Go is the type in which penultimate lines of ionization potential $\chi_1 = 7{\cdot}5$ volts attain their maximum; and we will take the excitation potential to be $\chi_1 - \chi_1^{(s)} = 3$ volts. These numbers are only rough. I have been unable to extract satisfactory values from the published data.* We assume the "weight" factor q_1/q_0 to be unity, and we take $\tau_0 = \frac{1}{3}$. Assuming the conditions are those of Problem I.,† we find from (65) and (58) or (66) the following numerical results :—

$$\kappa = \textit{constant} = \bar{\kappa}.$$

	Sun.	*Capella.*
K_1	$2{\cdot}08 \times 10^2$	$3{\cdot}41 \times 10$ dyne cm.$^{-2}$
P_0	$3{\cdot}96 \times 10^2$	$6{\cdot}40 \times 10$ dyne cm.$^{-2}$
$\bar{\kappa}$	5·88	0·817 cm.^{2} grm.$^{-1}$

These values for $\bar{\kappa}$ are much smaller than have been previously found for the photospheres of stars.‡ The present method is, however, the only direct observational method that has yet been given for the determination of the coefficient of opacity. Other methods have involved a physical theory of the absorption coefficient. It should be noted that our values refer not to the deeper photospheric layers, but to the upper part of the reversing layer, between $\tau = 0$ and $\tau = \frac{1}{3}$. The value 5·88 for the Sun agrees with the value 5·5 tentatively deduced in the preceding paper from Unsöld's published evaluations of N. The present values depend, however, not (as Unsöld's) on any theory of the atomic absorption coefficient but simply on the observed T_{max} for penultimate lines of ionization potential 7·5 volts and excitation potential 3 volts.

We notice that $\bar{\kappa}$ for Capella is some 7 times smaller than for the Sun, whilst the value of P_0 for Capella is some 6 times smaller than for the Sun. This suggests that $\kappa \propto P$ would be more in agreement with the facts. We therefore re-investigate our maximum problem on the basis $\kappa = \alpha P/(kT)^{3/2} = \alpha' P$.

* Since completing the paper I have noticed that Miss Payne (*Harvard Circular*, No. 307) finds that lines of I.P. 7·6 volts attain their maximum in stars of type Ko. The present calculations may therefore need revision—the values of P_0 would come out lower, and the values of $\bar{\kappa}$ higher. But until "maxima" are determined for constant r, it does not seem worth while re-calculating.

† Possibly the conditions correspond more closely to Problem III., but it is convenient to illustrate Problem I. fully.

‡ Milne, *M.N.*, **85**, 768, 1925; Eddington, *The Internal Constitution of the Stars* (1926), p. 359.

17. *Problem I., with* $\kappa = \alpha P/(kT)^{9/2} = \alpha' P$.—We have now to find the maximum of $N_0^{(s)}$ as given by (57), subject to P_0 being given by (55), with $\tau = \tau_0$, $P = P_0$, namely,

$$\tau_0 = \frac{\alpha'}{g} P_0^2 \left(1 + \frac{2}{3}\frac{P_0}{K_1}\right) \quad . \quad . \quad . \quad (67)$$

Differentiating (67) logarithmically we find

$$0 = \left(1 + \frac{2}{3}\frac{P}{K_1}\right)\frac{d\alpha'}{\alpha'} + \frac{2dP_0}{P_0}\left(1 + \frac{P_0}{K_1}\right) - \frac{2}{3}\frac{P_0}{K_1}\frac{dK_1}{K_1} \quad . \quad (68)$$

where $d\alpha'/\alpha' = -\frac{9}{2}dT/T$. It is useful to put λ for $\frac{9}{2}$ so as to trace the identity of this exponent. Eliminating dP_0 between (68) and (59) and solving for P_0, we find

$$P_0 = K_1\left[\frac{\chi_1^{(s)} + (\frac{5}{2} - \lambda)kT}{\frac{2}{3}\chi_1 - \chi_1^{(s)} + (\frac{2}{3}\lambda - \frac{5}{6})kT}\right]_{T=T_{max}} \quad . \quad . \quad (69)$$

with $\lambda = 9/2$.

We observe that unless $\frac{2}{3}\chi_1 > \chi_1^{(s)}$, *i.e.* unless

$$\chi_1 - \chi_1^{(s)} > \tfrac{1}{3}\chi_1 \quad . \quad . \quad . \quad . \quad (70)$$

the exponential factors alone do not cause $N_0^{(s)}$ to have a maximum; for the denominator in (69) might then be negative. Thus apart from the effect of powers of T, a necessary condition for a penultimate line to have a maximum when $\kappa = \alpha' P$ is

$$\text{E.P.} > \tfrac{1}{3}\text{I.P.} \quad . \quad . \quad . \quad . \quad (71)$$

If this condition is not satisfied the line should decrease in intensity with advancing temperature. We find in fact

$$\frac{dN_0^{(s)}}{N_0^{(s)}} = \frac{(P_0/K_1)[\frac{2}{3}\chi_1 - \chi_1^{(s)} + (\frac{2}{3}\lambda - \frac{5}{6})kT] - [\chi_1^{(s)} + (\frac{5}{2} - \lambda)kT]}{1 + P_0/K_1}\frac{dT}{kT^2};$$

lines with very small E.P.'s should therefore show no maximum. This is the first main difference between $\kappa = \alpha' P$ and $\kappa =$ constant $= \bar{\kappa}$. It arises from the circumstance that when $\kappa = \alpha' P$, P_0 increases less rapidly with T than when $\kappa =$ constant. It would be very interesting to know the actual behaviour of lines with low E.P.'s at low temperatures.

Provided $\chi_1 - \chi_1^{(s)}$ exceeds $\frac{1}{3}\chi_1$ appreciably, the numerical value of P_0 given by (69) does not differ much from that given by (65).

We can now find P_0 from (69) from an observed T_{max}, and inserting this in (67) determine α' and so α. The test of the theory will be that for different maxima in sequences of stars of constant g, $\alpha = \alpha'(kT)^{\frac{9}{2}}$ should come out always the same, independent of the lines chosen and of the g-value for the sequence.

If we denote by κ_0 the absorption coefficient at the level P_0, then $\kappa_0 = \alpha' P_0$, and hence by (67)

$$\tau_0 = \frac{\kappa_0 P_0}{g}\left(1 + \frac{2}{3}\frac{P_0}{K_1}\right) \quad . \quad . \quad . \quad (72)$$

Comparing this with (58) we see that the value of $\bar{\kappa}$ previously determined on the assumption κ = const. = $\bar{\kappa}$ is not very different from the value of κ at the level P_0 on the assumption $\kappa = \alpha P/(k)^{\frac{3}{2}}$.

18. *Numerical Values.*—We proceed to apply the formulæ of the last section to the Sun and Capella. We find the following values, using the same data as before :—

$$\kappa = \alpha P/(kT)^{\frac{3}{2}} = \alpha' P.$$

	Sun.	*Capella.*
K_1	$2{\cdot}08 \times 10^2$	$3{\cdot}41 \times 10$ dyne cm.$^{-2}$
P_0	$4{\cdot}65 \times 10^2$	$8{\cdot}35 \times 10$ dyne cm.$^{-2}$
κ_0	7·88	0·922 cm.^{2} grm.$^{-1}$
α'	0·0169	0·0110
α	$5{\cdot}7 \times 10^{-57}$	$2{\cdot}37 \times 10^{-57}$

The opacity coefficients κ_0 are of the same order as the previous values of $\bar{\kappa}$. But chief interest attaches to the values of α. These, of course, should be equal if the theory and data were correct. They are, however, very sensitive to the ionization potentials chosen, and we may well be content with the nearness of the two values to one another. α = constant suits our data better than $\bar{\kappa}$ = constant.

19. *Comparison with the theoretical Value of the Absorption Coefficient.*—I have shown in a previous paper * that the coefficient of general opacity, deduced from Kramers' formula by a method due in principle to Eddington, is

$$\kappa = \frac{1}{\bar{m}} \frac{80}{\pi^2\sqrt{3}} \frac{Z^2e^6h^2}{c(2\pi m_e)^{\frac{3}{2}}} \frac{P}{(kT)^{\frac{3}{2}}} f \quad . \quad . \quad . \quad (73)$$

Here m_e is the mass of the electron, $\bar{m}$ the mean weight of an atom of the gas, Z the number of charges on an ion,† and f a numerical factor. We have

$$\frac{80}{\pi^2\sqrt{3}} \frac{e^6h^2}{c(2\pi m_e)^{\frac{3}{2}}} = 1{\cdot}87 \times 10^{-79}.$$

We shall take Z = 1, corresponding to matter once ionized on the whole. The factor f is equal to unity when the opacity arises only from "free-free" transitions; when captures are taken into account it depends on the uncertain "Kramers' factor" involving the ionization potentials. We leave f undetermined for the moment. The only other factor in κ is $\bar{m}$, the mean atomic weight. Until the composition of the stellar atmosphere by mass is known, $\bar{m}$ is uncertain, but to obtain the order of magnitude we shall take $\bar{m}$ to correspond to calcium atoms, namely, $\bar{m} = 40 \times 1{\cdot}65 \times 10^{-24}$ grams.

* *M.N.*, **85**, 772, 1925.

† In a former application I took Z to be the nuclear charge. Eddington pointed out (*Internal Constitution of the Stars*, p. 359) that, since collisions with ions are in question, it is rather the ionic charge that should be taken.

Inserting numerical values in (73) and comparing with $\kappa = \alpha P(kT)^{9/2}$ we find the theoretical value of α is

$$\alpha = 2{\cdot}83 \times 10^{-57} f.$$

The values of α found observationally by the method of maxima in § 18 are $\alpha = 5{\cdot}7 \times 10^{-57}$ from the Sun and $\alpha = 2{\cdot}37 \times 10^{-57}$ from Capella. f being a numerical factor, it is a gratifying result of the method of the present paper that it gives values of α of the predicted order of magnitude. If we can trust even the first significant figure of the Sun and Capella values, they indicate that f is approximately unity. This would mean that "free-free" electronic encounters control the opacity. I hope to discuss the theoretical value of f on another occasion.

It should be noted that the coefficient of opacity required in the present investigation is not the "Rosseland mean"

$$\frac{1}{\kappa} = \int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu}{\partial T} d\nu \Big/ \int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu,$$

but the "straight" mean

$$\kappa = \int_0^\infty \kappa_\nu B_\nu d\nu \Big/ \int_0^\infty B_\nu d\nu.$$

Formula (73) is derived from the "straight" mean.

If the factor f could be taken as unity on theoretical grounds, formula (73) combined with an observational determination of α by the method of maxima would determine $\bar{m}$, the mean weight * of the atoms composing the reversing layer. Conversely, if $\bar{m}$ were known we could determine f observationally. $\bar{m}$ is in principle capable of direct determination from observation as follows: Once α is determined by an observed maximum, P_0 is determined by (67). Inserting this in (57) we determine $N_0^{(s)}/\epsilon$. An application of Unsöld's method then determines $N_0^{(s)}$ and so ϵ. If now ϵ be determined for all constituents present in the reversing layer, $\bar{m}$ can be calculated. There is here a considerable field for observational work.

This opens up the possibility of determining the physical constant $e^6h^2/c(2\pi m_e)^{3/2}$ by astrophysical methods.

So far we have outlined the method of maxima entirely on the basis of the formulæ arising in Problem I. Each of the other problems can be treated similarly, the only difference being in the algebraic formulæ. Each problem provides a determination of $\bar{\kappa}$ or κ_0 and α by the method of maxima. We shall now briefly obtain the formulæ for some of the other problems.

20. *Problem II.*—$\kappa = const. = \bar{\kappa}$. We have from (30′)

$$N_1^{(s)} = \frac{2\epsilon}{mg} A_1^{(s)} \left[\frac{P_0(K_2 + P_0)}{2K_2 + P_0} - K_2 \log\left(1 + \frac{1}{2}\frac{P_0}{K_2}\right)\right] \quad . \quad (74)$$

* Not "the mean molecular weight of the particles present," which reckons in the free electrons.

with

$$\tau_0 = \frac{\bar{\kappa}}{g}\frac{P_0(2P_0 + 3K_2)}{P_0 + 2K_2} \quad . \quad . \quad . \quad (75)$$

We have to make $N_1^{(s)}$ a maximum for varying T, P_0 being a function of T given by (75). It is simplest to replace P_0 by a variable θ, given by

$$\theta = P_0/K_2.$$

By logarithmic differentiation of (74) and (75) we find after a little reduction that T_{max} is the solution of the equation

$$\frac{\chi_2 - \chi_2^{(s)}}{\chi_2 + \frac{5}{2}kT} = \frac{\log(1 + \frac{1}{2}\theta) - \frac{\frac{1}{2}\theta}{\theta + 1}}{\frac{\theta(\theta+1)}{\theta+2} - \log(1 + \frac{1}{2}\theta)} \quad . \quad . \quad (76)$$

$$\equiv \psi_1(\theta),$$

say. It is easily shown that $\psi_1(0) = 1$, $\psi_1(\infty) = 0$, $0 < \psi_1(\theta) < 1$. The equation has always a single root. Numerical values of $\psi_1(\theta)$ are

$\theta =$	$\frac{1}{2}$,	1,	2.
$\psi_1(\theta) =$	0·73,	0·59,	0·445.

This shows (having regard to typical numerical values of χ_2 and $\chi_2^{(s)}$) that the root θ is in the neighbourhood of 2. Thus $P_0 = 2K_2$ at T_{max}, approximately. The theory of the present section could be applied, for example, to the maximum of the penultimate lines of Si^+, but we shall not go into the details here.

21. *Problem II.* $\kappa = \alpha P/(kT)^{9/2} = \alpha' P$.—We have to determine the maximum of (74) subject to (52) with $\tau = \tau_0$, $P = P_0$. Putting $P_0/K_2 = \theta$ as before, we find after some reduction that T_{max} is the root of the equation,

$$\frac{\chi_2^{(s)} + \frac{5}{2}kT}{\chi_2 + \frac{5}{2}kT} - \frac{\lambda kT}{\chi_2 + \frac{5}{2}kT}\frac{\frac{\theta(\frac{1}{2}\theta^2 + \theta + 1)}{\theta + 2} - \log(1 + \frac{1}{2}\theta)}{(\theta+1)\left[\frac{\theta(\theta+1)}{\theta+2} - \log(1 + \frac{1}{2}\theta)\right]}$$

$$= \frac{2\theta}{\theta + 1}\frac{\frac{1}{2}\theta - \log(1 + \frac{1}{2}\theta)}{\frac{\theta(\theta+1)}{\theta+2} - \log(1 + \frac{1}{2}\theta)} \quad . \quad (77)$$

$$\equiv \psi_2(\theta),$$

say, where $\lambda = \frac{9}{5}$. It is found that $\psi_2(0) = 0$, $\psi_2(\infty) = 1$, $0 < \psi_2(\theta) < 1$. Numerical values are

$\theta =$	1,	2.
$\psi(\theta) =$	0·362,	0·507.

Thus the values of θ, for usual values of χ_2 and $\chi_2^{(s)}$, is about 2, and $P_0 = 2K_2$.

The solutions of (76) and (77) could easily be found to any required degree of accuracy for given χ_2, $\chi_2^{(s)}$, and T_{max}. We notice that the formulæ are no longer of the simple Fowler-Milne type.

22. *Problem III.* $\kappa = \text{const} = \bar{\kappa}$.—We have here

$$N_0^{(s)} = \frac{2\epsilon}{mg} A_0^{(s)}\left[P_0 - K_1 \log\left(1 + \frac{P_0}{K_1}\right)\right] \quad . \quad . \quad (78)$$

with

$$\tau_0 = \frac{\bar{\kappa}}{g} 2P_0 \quad . \quad . \quad . \quad . \quad (79)$$

Hence in this problem P_0 is independent of T. Differentiating (78) with regard to T for a maximum, subject to (79) we find on putting $P_0/K_1 = \theta$ that T_{max} is given by

$$\frac{\chi_1 - \chi_1^{(s)}}{\chi_1 + \frac{5}{2}kT} = \frac{\log(1+\theta) - \dfrac{\theta}{1+\theta}}{\theta - \log(1+\theta)} \quad . \quad . \quad (80)$$

$$\equiv \psi_3(\theta),$$

say. We have $\psi_3(0) = 1$, $\psi_3(\infty) = 0$, $0 < \psi_3(\theta) < 1$. A numerical value is $\psi_3(1) = 0{\cdot}63$.

We apply the theory to the Balmer lines of hydrogen, with $\chi_1 = 13{\cdot}54$ volts, $\chi_1^{(s)} = 3{\cdot}385$ volts. For $T_{max} = 10{,}000°$, the left-hand side of (80) is 0·65. Hence the root is $\theta = 1$, approximately, or $P_0 = K_1$. We require the average value of g for stars of type A_0, but to be definite we shall adopt the value of g for Sirius. Adams and Russell * give for this star $g = 0{\cdot}38\odot = 1{\cdot}04 \times 10^4$ cm. sec.$^{-2}$. Putting $\tau_0 = \frac{1}{3}$ in (79) we then find as follows :—

Sirius.	*Balmer Lines.*	κ = *constant* = $\bar{\kappa}$.
T_{max}	10,000°	11,000°
K_1	$4{\cdot}9 \times 10^2$	$2{\cdot}9 \times 10^3$
P_0	$4{\cdot}9 \times 10^2$	$2{\cdot}9 \times 10^3$ dyne cm.$^{-2}$
$\bar{\kappa}$	3·5	0·61 cm.^{2} gram.$^{-1}$

The values of $\bar{\kappa}$ are a little smaller than those found for the Sun and Capella, but of the same order.

We note in passing that the ratio of the force due to *general* radiation-pressure to that due to gravity is

$$\bar{\kappa}\sigma T^4/cg = 3{\cdot}5 \times 5{\cdot}7 \times 10^{-5} \times 10^{16}/3 \times 10^{10} \times 1{\cdot}04 \times 10^4 = 6 \times 10^{-3},$$

so that our neglect of *general* radiation-pressure is amply justified.

23. *Problem III., with* $\kappa = \alpha P/(kT)^{\frac{1}{2}} = \alpha' P$.—We have now to discuss the maximum of (78) subject to

$$\tau_0 = \frac{\alpha'}{g} P_0^2 \quad . \quad . \quad . \quad . \quad (81)$$

* *Astrophys. Journ.*, **68** (1928), p. 33, table xii.

We see that P_0 is no longer constant, since α' involves T. We find on carrying out the differentiations

$$\frac{\chi_1 - \chi_1^{(s)}}{\chi_1 + \frac{5}{2}kT} + \frac{\frac{1}{2}\lambda kT}{\chi_1 + \frac{5}{2}kT}\frac{\theta^2/(1+\theta)}{\theta - \log(1+\theta)} = \frac{\log(1+\theta) - \theta/(1+\theta)}{\theta - \log(1+\theta)} \quad (82)$$

$$\equiv \psi_3(\theta),$$

with $\lambda = \frac{9}{2}$, $\theta = P_0/K_1$. The term in λ is unimportant numerically; and to save trouble we shall take the root to be $\theta = 1$ for the Balmer lines of hydrogen, as before. We then find as follows, α' being determined from (81):—

Balmer Lines.	*Sirius.*	$\kappa = \alpha P/(kT)^{5/2} = \alpha' P.$
T_{max}	10,000°	11,000°
$P_0(=K_1)$	$4{\cdot}9\times10^2$	$2{\cdot}9\times10^3$ dyne cm.$^{-2}$
κ_0	7·0	1·21 cm.^{2} grm.$^{-1}$
α'	$1{\cdot}4\times10^{-2}$	$0{\cdot}42\times10^{-3}$
α	59×10^{-57}	$2{\cdot}7\times10^{-57}$

(κ_0 is the value of κ at the level P_0.)

We see that for $T_{max} = 11{,}000°$ the value of the constant α is very close to the values found for the Sun and Capella. Russell, Dugan, and Stewart (*Astronomy*, ii, p. 753) adopt the value T = 11,000° for stars of type Ao.

There is a fundamental maximum problem associated with Problem IV., but the calculations are rather complicated, and I have considered it not worth while to cumber the present paper with the results. The obtaining of them is only a matter of algebra. When detailed observations have been made on low temperature maxima, it will probably be worth while analysing them with regard to the formulæ of this problem, and attempting to disentangle the effects of χ_1 and $\overline{\chi}_1$. In the meantime we proceed to Problem V., the problem of the behaviour of the principal lines of an ionized atom.

24. *Problem V., with κ = constant = $\bar{\kappa}$.*—We have

$$N_1 = \frac{2\epsilon}{mg}\left[\frac{P_0(P_0+K_2)}{P_0+2K_2} - K_2\log\left(1+\frac{1}{2}\frac{P_0}{K_2}\right)\right] \quad . \quad (83)$$

$$\tau_0 = \frac{\bar{\kappa}}{g}P_0\frac{3K_2+2P_0+P_0^2/K_1}{2K_2+P_0} \quad . \quad . \quad . \quad . \quad (84)$$

Put

$$P_0 = K_2\phi.$$

Then

$$N_1 = \frac{2\epsilon}{mg}K_2\left[\frac{\phi(\phi+1)}{\phi+2} - \log(1+\tfrac{1}{2}\phi)\right] \quad . \quad . \quad (85)$$

and

$$\tau_0 = \frac{\kappa}{g}K_2\phi\frac{3+2\phi+(K_2/K_1)\phi^2}{2+\phi} \quad . \quad . \quad . \quad (86)$$

We have to make (85) a maximum subject to (86). It is convenient to use the relation

$$d\left(\frac{K_2}{K_1}\right) = \frac{K_2}{K_1}\frac{\chi_2 - \chi_1}{kT^2}dT.$$

We find then after some reduction,

$$\frac{\frac{1}{2}\phi(1 + (K_2/K_1)\phi^2)}{1 + \phi + (K_2/K_1)\phi^2} - \log(1 + \tfrac{1}{2}\phi) = \frac{\chi_2 - \chi_1}{\chi_2 + \frac{5}{2}kT}\frac{(K_2/K_1)\phi^4}{2(2 + \phi)(1 + \phi + (K_2/K_1)\phi^2)} \quad (87)$$

Now write $P_0 = \theta(K_1K_2)^{\frac{1}{2}}$, so that $\phi = \beta\theta$, where $\beta = (K_1/K_2)^{\frac{1}{2}}$. Then (87) becomes

$$\frac{\frac{1}{2}\beta\theta(1 + \theta^2)}{1 + \beta\theta + \theta^2} - \log(1 + \tfrac{1}{2}\beta\theta) = \frac{\chi_2 - \chi_1}{\chi_2 + \frac{5}{2}kT}\frac{\beta^2\theta^4}{2(2 + \beta\theta)(1 + \beta\theta + \theta^2)} \quad (88)$$

The left-hand side of (88) can be written in the form

$$\tfrac{1}{2}\beta^2\int_0^{\theta}\frac{\theta(\theta^2 - 1)(3 + 2\beta\theta + \theta^2)}{(2 + \beta\theta)(1 + \beta\theta + \theta^2)^2}d\theta,$$

which shows that the root θ of (88) must be at least greater than unity. Since β is large, it follows that $\beta\theta$ is large compared with unity. Assuming for the moment that $\beta\theta$ is also large compared with θ^2, (88) may be approximated to in the form

$$\tfrac{1}{2}(1 + \theta^2) - \log(1 + \tfrac{1}{2}\beta\theta) = \frac{\chi_2 - \chi_1}{\chi_2 + \frac{5}{2}kT}\tfrac{1}{2}\theta^2,$$

which gives

$$\theta^2 = \frac{\chi_2 + \frac{5}{2}kT}{\chi_1 + \frac{5}{2}kT}\left[2\log(\tfrac{1}{2}\beta\theta) - 1\right] \quad . \quad . \quad (89)$$

Now apart from a small term depending on the statistical weights of the normal states of the atoms, $\log\beta = \frac{1}{2}(\chi_2 - \chi_1)/kT$. Hence θ is the root of the equation

$$\theta^2 = \frac{\chi_2 + \frac{5}{2}kT}{\chi_1 + \frac{5}{2}kT}\left[\frac{\chi_2 - \chi_1}{kT} + \log\theta^2 - 1 - 2\log 2\right] \quad . \quad (90)$$

We verify that β is large compared with θ, as assumed.

We have now that P_0 is given by

$$P_0 = \theta(K_1K_2)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (91)$$

which by (90) gives approximately

$$P_0 = \left[K_1K_2\frac{\chi_2 + \frac{5}{2}kT}{\chi_1 + \frac{5}{2}kT}\frac{\chi_2 - \chi_1}{kT}\right]^{\frac{1}{2}}_{T=T_{max}} \quad . \quad . \quad (92)$$

The earlier Fowler-Milne formula was

$$\bar{P}_e = \left[K_1K_2\frac{\chi_2 + \frac{5}{2}kT}{\chi_1 + \frac{5}{2}kT}\right]^{\frac{1}{2}}_{T=T_{max}} \quad . \quad . \quad . \quad (92')$$

We notice at once an important difference in the occurrence of the factor $[(\chi_2 - \chi_1)/kT]^{\frac{1}{2}}$ on the right-hand side of (92). This factor is considerable at low temperatures, and its occurrence helps to remove the existing anomaly of the very low values previously found for $\bar{P}_e$ from the maxima of Ca^+ and Sr^+ lines. It does not, however, remove the anomaly by any means.

Formula (84) now gives

$$\tau_0 = \frac{\bar{\kappa}}{g} P_0 \frac{3 + 2\beta\theta + \theta^2}{2 + \beta\theta},$$

which gives approximately

$$\tau_0 = \frac{\bar{\kappa}}{g} 2 P_0 . \quad . \quad . \quad . \quad . \quad (93)$$

The H and K lines of Ca^+ are so wide near their maximum that until the maximum is re-determined in the precise sense of the present paper (*i.e.* for given r) it is difficult to know how to make accurate numerical application of the foregoing formulæ. Let us, however, assume tentatively that the H and K lines have their maximum in Arcturus, of type Ko. The temperature adopted for this star by Russell, Dugan, and Stewart is 4200°; we will work out the results for T = 4200° and T = 5000°. The value of g for this star (Adams and Russell, *loc. cit.*) is $0{\cdot}012\odot = 0{\cdot}012 \times 2{\cdot}74 \times 10^4 = 3{\cdot}3 \times 10^2$. We then find the following ($\chi_1 = 6{\cdot}08$ volts, $\chi_2 = 11{\cdot}82$ volts):—

T	=	4200°	5000°
θ	=	5·55	5·06
$(K_1K_2)^{\frac{1}{2}}$	=	$0{\cdot}661 \times 10^{-2}$	0·533
P_0	=	$3{\cdot}67 \times 10^{-2}$	2·70 dyne cm.$^{-2}$
$\bar{\kappa}$	=	$1{\cdot}49 \times 10^3$	20·3 gram.$^{-1}$ cm.^{2}

The value of $\bar{\kappa}$ should be of the same order of magnitude as found previously for the Sun, Capella, and Sirius. We deduce either that there is something radically wrong with the theory, or else that the maximum of the H and K lines (in the precise sense of this paper) must be at 5000° or a little higher. Until the maximum is properly determined, it is idle to discuss the matter further. The H and K lines are, of course, chromospheric lines, the Ca^+ atoms are exposed to strong selective radiation-pressure, and probably we ought to take a much smaller value of g. Equation (93) shows that this would tend in the direction of reducing the discrepancy.

In order to be sure that the special conditions of Problem V. have little effect, we now discuss the maximum problem for Problem V*a*., in which the atoms considered are surrounded with atoms already once ionized, so that $P = \frac{1}{2}p$.

25. *Problem Va., with κ = constant = $\bar{\kappa}$.*—We have here, from (52″),

$$N_1 = \frac{2\epsilon}{mg}\left[P_0 - K_2 \log\left(1 + \frac{P_0}{K_2}\right) - \frac{1}{2}\frac{P_0^2}{K_1}\right] . \quad . \quad (94)$$

and from (51)

$$\tau_0 = \frac{\bar{\kappa}}{g} 2P_0 \quad . \quad . \quad . \quad . \quad (95)$$

Here P_0 is independent of T. Differentiating (94) with regard to T, we have at the maximum

$$\frac{dK_2}{K_2}\left[\log\left(1+\frac{P}{K_2}\right) - \frac{P_0/K_2}{1+P/K_2}\right] = \frac{\frac{1}{2}P_0^2}{K_1K_2}\frac{dK_1}{K_1},$$

which on putting $P_0 = \theta(K_1K_2)^{\frac{1}{2}}$ gives

$$\log(1+\beta\theta) - \frac{\beta\theta}{1+\beta\theta} = \frac{\frac{1}{2}\theta^2\chi_1 + 5/2kT}{\chi_2 + 5/2kT},$$

or, since $\beta\theta$ is evidently large compared with unity,

$$\theta^2 = \frac{\chi_2 + \frac{5}{2}kT}{\chi_1 + \frac{5}{2}kT}[2\log\beta\theta - 2],$$

or again,

$$\theta^2 = \frac{\chi_2 + \frac{5}{2}kT}{\chi_1 + \frac{5}{2}kT}\left[\frac{\chi_2 - \chi_1}{kT} + \log\theta^2 - 2\right] \quad . \quad . \quad (96)$$

The value of θ at T_{max} is thus substantially the same as that given by (90), and so Problem V*a*. gives the same value for P_0, and hence for $\bar{\kappa}$ (by (93)), as Problem V. We therefore give no separate numerical discussion.

26. *Problem V., with* $\kappa = \alpha' P$.—The equations are (83) and

$$\tau_0 = \frac{\alpha'}{g}P_0^2\left(1+\frac{2}{3}\frac{P_0}{K_1}\right) \quad . \quad . \quad . \quad (97)$$

We find after some rather tedious algebra that N_1 is a maximum when

$$\frac{\chi_2 + \frac{5}{2}kT}{\chi_1 + \frac{5}{2}kT}\frac{(\beta\theta+2)^2}{\beta\theta(\beta\theta+3)}\left[2\log(1+\tfrac{1}{2}\beta\theta) - \frac{4\beta\theta}{(2+\beta\theta)^2}\right] - \frac{\frac{2}{3}\theta^2}{1+\theta/\beta} = \frac{\lambda kT}{\chi_1 + \frac{5}{2}kT}\frac{1+\frac{2}{3}\theta/\beta}{1+\theta/\beta}\beta\theta \quad . \quad (98)$$

where, as usual, $P_0 = \theta(K_1K_2)^{\frac{1}{2}}$, $\beta = (K_1/K_2)^{\frac{1}{2}}$, and $\lambda = \frac{2}{3}$.

This result discloses a new and somewhat disconcerting feature. If λ were zero, the result would be practically that for Problem V. or V*a*. with $\kappa = \bar{\kappa}$, the only difference being the occurrence of $\frac{2}{3}\theta^2$ instead of θ^2. As it is, the term in λkT happens to have for a factor the large coefficient β. The term in λkT is thus dominant to the one in θ^2, and for $\beta\theta$ large compared with unity, (98) yields approximately

$$\theta = \frac{1}{\beta}\frac{\chi_2 + \frac{5}{2}kT}{\lambda kT}\left[\frac{\chi_2 - \chi_1}{kT} + \log\theta^2 - 1 - 2\log 2\right] \quad . \quad (99)$$

and

$$P_0 = K_2\frac{\chi_2 + \frac{5}{2}kT}{\lambda kT}\left[\frac{\chi_2 - \chi_1}{kT} + \log\theta^2 - 1 - 2\log 2\right]. \quad (100)$$

Thus in spite of the small factor kT in the denominator, P_0 at T_{max}

is now substantially of the order of K_2, *i.e.* much smaller than before. The apparently innocuous exponent $\lambda = \frac{9}{2}$ in our expression for κ has completely changed the solution of the problem.

The reason of this is that the maximum of N_1 in Problem V. is so flat that the introduction of a relatively slowly varying factor like $T^{-\frac{9}{2}}$ in the expression for τ_0 completely alters the exact location of the maximum. It is easily found, in fact, that N_1 has on the face of it not only a maximum but a minimum, since the presence of $T^{-9/2}$ in a' makes P_0 ultimately increase, and so makes N_1 ultimately increase again. It is clear that when this occurs we have got beyond the region where the conditions of Problem V. hold.

It would be wrong to minimise the difficulty caused by this appearance of the λ-term as dominant. It is of course useless to test (100) observationally—the observed T_{max}, if at the usually accepted temperature, would give very small values of P_0 and very large values for κ_0 or a. At the same time the fact appears to be that for a flat maximum like that of the principal lines of the ionized atom, the exact position of the maximum is very sensitive to the precise formulæ or assumptions adopted, and accurate conclusions cannot be drawn unless great refinement is introduced.

27. Miss Payne has recently * (*Harv. Repr.*, **48**, 405) published a curve of values of N_1, derived by Unsöld's method for Ca^+, against T, and states that the observed points show no maximum. It is difficult to see how there can fail to be a maximum somewhere or other, on physical grounds, and our exact formulæ (49″) and (52″) in conjunction with the formulæ for τ_0 certainly give values of N_1 which at low temperatures diminish as T diminishes. Actually Miss Payne's plotted points for giant stars seem not inconsistent with a maximum between Go and Ko, which would give a very fair value for $\bar{\kappa}$, in the light of §24. All our calculations of maxima, of course, are based on constant g, so that sequences of stars of the same g should alone be considered.

28. *Calculation of an absolute Abundance.*—Since there appear to be difficulties in the application of the method of maxima to the H and K lines of Ca^+, it is desirable to check whether our formulæ are substantially on the right lines. We therefore make a direct calculation of N_1/ϵ for Ca^+ atoms on the Sun.

We have already found from an observed maximum the value $P_0 = 4{\cdot}65 \times 10^2$ dyne cm.$^{-2}$ at the depth $\tau_0 = \frac{1}{3}$ on the Sun. This should be independent of the particular kind of atom, and we can therefore use it for Ca^+ atoms. For Ca^+ atoms the value of K_2 at 5740° is $3{\cdot}30 \times 10^{-2}$ dyne cm.$^{-2}$. This is so small compared with P_0 that (83) reduces to

$$\frac{N_1}{\epsilon} = \frac{2}{mg} P_0.$$

Inserting the above value of P_0, with $g = 2{\cdot}74 \times 10^4$, $m = 40 \times 1{\cdot}64 \times 10^{-24}$ we find

$$N_1/\epsilon = 0{\cdot}52 \times 10^{21}.$$

* Published in *Proc. Nat. Acad. Sci.*, **14**, 399, 1928.

Now Unsöld has found for N_1 for Ca^+ atoms, by line-contours, the value

$$N_1 = 2{\cdot}33 \times 10^{19}.$$

We deduce

$$\epsilon = 4{\cdot}5 \times 10^{-2}.$$

Thus according to this calcium constitutes about 4·5 per cent. by mass of the solar reversing layer—a very reasonable value.

This paragraph illustrates the general method by a very simple case. In other cases P_0 must be calculated from the appropriate formula, using the numerical value of $\bar{\kappa}$ or α for given T and g. $N_r^{(s)}/\epsilon$ is then found from this value of P_0, again for the given T and g, and compared with the value of $N_r^{(s)}$ found by Unsöld's method. We thus deduce ϵ.

28*a*. *Direct Determination of P_0 by Unsöld's Method without using Maxima.*—A method of determining P_0 entirely independent of the method of maxima, and probably ultimately an exceedingly accurate method, has recently been employed in principle by Payne and Hogg,* though they used only the crude Saha formulæ. Suppose N_0 and N_1, *e.g.* for Ca and Ca^+, are found by Unsöld's method. Let these values be inserted in equation (50′) and the equation solved for P, which is here P_0. This gives directly from observation the electron-pressure at level τ_0. The value of $\bar{\kappa}$ or of α can then be computed from the corresponding equation for τ_0, *i.e.* (84) or (97). It is not proposed to apply this method to an actual case in the present paper, but Payne and Hogg's results are precisely in the form for immediate application, assuming they refer substantially to points on the line-contour of constant r. It should be noted that since the ratio N_1/N_0 is alone involved in (50′), the actual value of the constant used in Unsöld's formula for the atomic absorption coefficient is not involved save in a certain factor f.† The application of this method offers an extensive field of work. With suitable modification of detail and use of the formulæ appropriate to the particular problem, it may be applied to any pair of lines, not necessarily to a pair of the same atom.

Payne and Hogg's actual values of the electron-pressure are somewhat smaller than those found in §§ 16, 18 above. Possibly the value $\chi_1 = 7{\cdot}5$ volts there taken is too small. Smaller values of P_0 would yield larger values of $\bar{\kappa}$ and τ.

29. *Concluding Remarks.*—No searching test of the theory expounded in this paper can be made until accurate determination of temperatures of "maximum intensity," in the strict sense above defined, have been carried out. Until this is done, numerical application must be regarded as provisional. It is clear that either determinations of maximum must be made for sequences of stars of constant g, or else—as, for example, on the "main sequence"—g must be considered as a function of T and suitable modifications made in the derivation of the maximum formulæ. Nevertheless such numerical applications as we have been able to make

* *Harvard Circular*, No. 334, 1928.

† Unsöld's notation—not the f of § 19.

disclose nothing at variance with the theory, and they lead to acceptable values of the mean coefficient of opacity $\bar{\kappa}$ and of the constant α in the formula $\kappa = \alpha P/(kT)^{9/2}$.

The possibility of thus determining α straight from observations of maxima would appear to be of considerable importance, in view of the discrepancy found by Eddington between the theoretical value of this constant and that deduced from the observed luminosities of stars. In the case of the interior of a star, we are dealing with matter at a temperature of several millions of degrees; in the case of a stellar atmosphere the temperature is only some thousands of degrees. It will be a matter of great interest if the absorption coefficient as deduced from Kramers' formula is found to have the right value for the stellar atmosphere, though the wrong value for the stellar interior. As far as the estimates of this paper go, observed and calculated opacities agree for the stellar atmosphere, but these estimates are only provisional.

30. *The Physical Argument.*—It is interesting to see, by purely physical arguments, why observation of maxima (in the strict sense) determines the intrinsic opacity of a stellar atmosphere. Ionization changes with pressure, and so the observed maximum can be fitted to the theoretical maximum by suitably choosing the electron-pressure P. Thus P is determined. The blackness of the line—in other words, the intensity-ratio at the point of the line-contour considered—determines the "total fogginess" of the stellar atmosphere down to level P; (r determines τ_0). But the total number of electrons down to level P is determined by P. Hence, dividing the "total fogginess" by the number of electrons, we obtain the "intrinsic fogginess per electron." This is simply the coefficient of general opacity—the constant α.

31. *Summary.*—(1) "Generalised Saha formulæ" are found, giving the number of atoms N_r in any given state of ionization, per cm.^{2} of a stellar atmosphere, down to the level where the electron-pressure is P, as a function of P, T, and the appropriate ionization potentials. Changes of ionization from level to level with changing pressure are accurately allowed for.

(2) The optical depth τ down to level P is calculated on two assumptions regarding the coefficient of general opacity of the stellar atmosphere, (*a*) κ = constant = $\bar{\kappa}$, (*b*) $\kappa = \alpha P/(kT)^{9/2}$ as given by Kramers' formula.

(3) The following is the definition of the maximum of an absorption line required in this paper. Let Q be an arbitrary point on the contour of a line, of frequency ν, and let Q_0 be the line-centre, of frequency ν_0. Let r be the intensity-ratio at Q. The temperature of maximum of the absorption line with reference to Q is that temperature at which $|\nu - \nu_0|$ is greatest for constant r.

(4) To a given r corresponds a definite value τ_0 of τ. The line reaches its maximum intensity, in the above precise sense, when $N_r^{(s)}$, the number of atoms in the appropriate state of ionization and excitation down to level τ_0, is a maximum subject to the condition τ_0 = constant.

(5) The above calculations and definitions enable us to dispense with the vague concept of the "mean pressure in a stellar atmosphere"

and to make calculations of maximum intensity based on definite pressures at definite levels. The equation τ_0 = constant defines the electron-pressure P_0 at level τ_0 as a function of T and g, and this function must be introduced into the expression for $N_r^{(s)}$ when differentiating for a maximum.

(6) The condition for the maximum of an absorption line when found in this manner is an equation connecting P_0 with T_{max}. If κ is supposed known, either on assumption (*a*) or assumption (*b*), the equation τ_0 = constant combined with the maximum condition determines T_{max} absolutely, without any empirical assumption as to the mean pressure in the stellar atmosphere as in former theories. Conversely, an observed value of T_{max} for a given line in a sequence of stars of given g determines on assumption (*a*) the mean value of the coefficient of absorption of the stellar atmosphere, and on assumption (*b*) the value κ_0 of the coefficient of opacity at level τ_0 and hence the value of the universal constant α.

(7) Assuming lines of I.P. 7·5 volts attain their maximum in stars of type Go, the analysis gives for the Sun (T = 5740°) the value $\bar{\kappa}$ = 5·9, and for Capella (T = 5200°) $\bar{\kappa}$ = 0·82. The electron-pressures at level τ_0 are $4{\cdot}0 \times 10^{-4}$ atmos. and $6{\cdot}4 \times 10^{-5}$ atmos. respectively when κ = constant = $\bar{\kappa}$; when $\kappa = \alpha P/(kT)^{9/2}$, the values of P_0 are $4{\cdot}65 \times 10^{-4}$ atmos. and $8{\cdot}35 \times 10^{-5}$ atmos. respectively. In the latter case we have for the absorption coefficient at level τ_0 the values κ_0 = 7·9 and κ_0 = 0·92 respectively, and the values deduced for the universal constant α are $5{\cdot}7 \times 10^{-57}$ and $2{\cdot}37 \times 10^{-57}$.

(8) Assuming the Balmer lines attain their maximum in stars of type Ao, the analysis gives for Sirius (T = 11,000°) the values $P_0 = 2{\cdot}9 \times 10^{-3}$ atmos., $\bar{\kappa}$ = 0·61, κ_0 = 1·2, $\alpha = 2{\cdot}7 \times 10^{-57}$.

(9) The foregoing astrophysical determinations of α may be compared with the theoretical value deduced from Kramers' formula. This gives for once-ionized matter of mean atomic weight $\bar{m}$ the value $\alpha = 1{\cdot}87 \times 10^{-79} f/\bar{m}$, where f is a numerical constant, probably of the order of unity. f is exactly unity if "free-free" transitions control the opacity. If we take f = 1, and adopt for $\bar{m}$ the atomic weight of calcium, $\bar{m} = 40 \times 1{\cdot}64 \times 10^{-24}$ grm. we have $\alpha = 2{\cdot}83 \times 10^{-57}$, which is of the order found astrophysically. The mean atomic weight can be found independently by use of (11) below applied to all the constituents of the stellar atmosphere, and hence Kramers' formula for the stellar absorption coefficient is capable of direct test.

(10) For the maximum of the principal lines of an ionized atom (like those of Ca^+ and Sr^+) the present method gives a revised formula which helps to remove the existing anomalies, but the numerical application still involves discrepancies. For various reasons it is not profitable to discuss these until precise determinations of maxima are made.

(11) Once $\bar{\kappa}$ or α has been determined from observations of maxima, the present method combined with an observational determination of $N_r^{(s)}$ by Unsöld's method (from a line-contour) affords a determination of ϵ, the absolute abundance (by mass) of the element in the stellar atmosphere. Unsöld's determination for ionized calcium atoms

on the Sun, $N_r^{(s)} = 2{\cdot}33 \times 10^{19}$ atmos. cm.$^{-2}$ leads to the value $\epsilon = 4{\cdot}5$ per cent. as the abundance of calcium by mass in the solar reversing layer. By finding ϵ in this way separately for all the constituents of the reversing layer, the mean atomic weight $\overline{m}$ could be found. The fact that $\Sigma\epsilon$ for all constituents should equal unity should give a check on the whole theory, both the present and Unsöld's.

(12) The formulæ of the present paper can be employed to determine P_0 directly from line-contour observations of N_1/N_0—a development of a method recently used by Payne and Hogg.

(13) If $\kappa = \alpha P/(kT)^{3/2}$, a necessary condition for penultimate lines in low-temperature stars to have a maximum is E.P.$>\frac{1}{3}$I.P.

(14) The formulæ can also be used to determine whether absorption lines for given stellar temperature should increase or decrease in intensity with change of g—that is, from dwarfs to giants. Discussion of this application is postponed to a later paper.

Viscosity in the Stars. By S. Rosseland.

(Communicated by Professor A. S. Eddington.)

A perusal of the observational data concerning the physical state of the stellar atmospheres is sufficient to show that a thorough understanding of these phenomena will involve an intimate knowledge of hydrodynamics in its various aspects. The first crude progress was made at an early stage of dynamical astronomy by investigators working on the problem of fluid gravitating masses in uniform rotation. The range of these early researches seems, however, to be seriously limited by the simplifying assumptions on which the mathematical investigations were based, a remark which would seem to concern some modern investigations in the same field.

A case of this kind is the question of viscous stresses in the stars. One will generally find that this problem is treated by considering some ideal large-scale motion involving skew stresses, such as a symmetrical pulsation of a non-rotating star,* or a purely rotational motion with angular velocity a function of the distance from the axis, or a function of the distance from the stellar centre, or both.† Such large-scale motions are necessarily attended by vanishingly small viscous stresses. These latter are proportional to the space variations of the velocity components, and for motions such as those mentioned above the variations are negligibly small. Thus Wilsing arrived at the result that the sun's equatoreal acceleration might, for aught we know, be an evanescent relic from some primordial state of motion of the

* E. Persico, *M.N.*, **86**, 98, 1925.

† *Cf.* T. Wilsing, *Astr. Nach.*, **127**, 233, 1891; E. T. Wilczynski, *Dissertation*, Berlin, 1897; R. A. Sampson, *Mem. R.A.S.*, **51**, 123, 1894; J. H. Jeans, *Astronomy and Cosmogony*, p. 271.

sun, which is still not smoothed out by viscosity. Jeans, on the other hand, finds that the viscous stresses in the sun are so small that the motion of the outer 26/27th part of the sun may, if other agencies be absent, be entirely governed by the braking effect of radiation-pressure, the necessary existence of which was first clearly pointed out by Jeans himself.

It may be doubted whether these conclusions are to be trusted, and for the following reasons :—All experience from laboratory experiments, and still more from meteorology and oceanography, shows that hydrodynamic motions of such an ideally simple kind as one is tempted to adopt in mathematical discussions of hydrodynamical problems are hard to produce and maintain. A linear flow in water in a tube can only be obtained for velocities below a critical limit, above which the laminar flow suddenly changes into turbulent motion of bewildering complexity. At this limit the viscous resistance to motion also experiences a sharp increase, a fact which immediately follows from the altered kinematical nature of the motion. The critical velocity is found to be inversely proportional to the cross-section of the tube, and for this reason the state of the atmosphere, the oceans, and still more the sun and the stars must always be that of turbulent motion, provided there exists a relative motion of the particles at all. A significant example of the importance of eddies for the resistance to shearing motion is offered by the fact that, neglecting the eddy-motion in the hydrodynamical equations, it is necessary to introduce a viscosity coefficient, which for the case of the atmosphere is about *half a million times* larger than the one found by laboratory experiments, in order to make theory and observation agree.*

The eddies are not only of importance for the transfer of momentum, as they will play a corresponding part in diffusion and temperature conduction besides viscosity. For all these phenomena it is an observational fact that the transfer by irregular eddy-motion far exceeds the transfer by random molecular motion as far as conditions in the atmosphere and in the oceans are concerned. For the corresponding problems in the stars we must evidently also take the eddies into account, and doing so we shall often expect to arrive at results which are very different from the predictions based on laminar motions.

If one has a doubt that eddy-motion plays any part in solar phenomena one need only take a glance at the sun in a good telescope to have the doubt removed. As all know, the surface of the sun presents a curious mottled aspect, the granulation, which changes rapidly from instant to instant, a typical feature of turbulent motion. We may perhaps even go further and take the magnificent eruptive prominence of 1919 May 29 as a cosmic illustration of the transition of laminar to turbulent motion. In fact, the lower base of the prominence was straight and rod-like while its upper part spread out into a fan-like

* *Cf.*, for instance, L. F. Richardson, *Weather Prediction by Numerical Process*, Cambridge University Press, 1922, p. 72. For a special monograph on exchange by eddies in general, *cf.* Wilhelm Schmidt, *Der Massenaustausch in freier Luft und verwandte Erscheinungen* (Probleme der kosmischen Physik VII.), Hamburg, 1925.

shape with a distinctly turbulent kind of motion, thus imitating the behaviour of a jet from a water-pipe scattering in the air.

Rotating Stars.—Having these considerations in mind we shall consider the theory of rotating stars. At present there are two main views prevailing of the agencies determining the state of rotation of a star. There is the view of Eddington,* according to which the rotational motion is strongly influenced by circulating convection currents, the intensity of which is determined by the distribution of the (unknown) heating sources in the star, an idea which took its origin in the theorem of von Zeipel. There is secondly the view of Jeans,† the starting-point of which is the braking effect of radiation. Jeans neglects explicitly all effects of convection currents, and believes the rotational state of the stars to be enforced by the braking effect alone. It has thus far been difficult to decide between these two views, but we shall see that by considering the effect of eddy-viscosity the importance of the braking effect of radiation would seem to be reduced below observational control, so that we are thrown back on the consideration of circulating currents as the main controlling factor of stellar rotation.

Consider first the theory of Jeans. We have then to distinguish between two main parts of the star, firstly a region near the stellar centre, where radiation is generated, and which we shall call the stellar core; and second, the part outside this region, where radiation is merely transmitted. Since the angular momentum of radiation moving in free space is conserved, there must exist a tendency for the emerging radiation to preserve the angular momentum it had at the moment of generation in the core. This tendency will go to retard or accelerate the motion of the masses outside the core, except for the case when the angular momentum of matter along a stream line of the flux of radiation is a constant. If the star is not too strongly flattened by rotation, this means, practically, that Wr^2 is a function of the latitude, as long as we remain outside the core, W being the angular velocity of rotation, and r the distance from the stellar centre. Consider as an idealised approximation the core to be spheroidal and to rotate with constant angular velocity ϖ. Let the meridians of the core be given by the equation of the ellipse

$$r^2 = \frac{a^2(1-e^2)}{1-e^2\cos^2\lambda},$$

a and e being major axis and excentricity, and λ the latitude. At the stellar surface we have then

$$W = \varpi\frac{r^2}{R^2} = \varpi\,.\,\frac{a^2}{R^2}\,\frac{(1-e^2)}{1-e^2\cos^2\lambda}$$
$$= \varpi\frac{a^2}{R^2}\left\{1-\frac{e^2}{1-e^2}\sin^2\lambda+\frac{e^4}{(1-e^2)^2}\sin^4\lambda-\ldots\right\}$$

* *The Internal Constitution of the Stars*, p. 284.
† *Astronomy and Cosmogony*, p. 266.

where R is the radius of the star, an expression which is of the right form to suit the observed rotation of the sun. In spite of this success the theory involves several points which are obscure. Thus it gives in its application to solar rotation the excentricity of the core as $e = 0{\cdot}36$. By the theory of rotating homogeneous masses we infer that the angular velocity of the core then must be

$$W = \sqrt{0{\cdot}035 \,.\, 2\pi\gamma\rho},$$

γ being the constant of gravitation and ρ the density of the core. Eddington's theory of the density distribution in the sun gives $\rho = 76$ g.cm.$^{-3}$, which finally fixes the rotation period of the core to 5/3 hours. This makes in turn the major axis of the meridian ellipse of the core equal to $\frac{1}{20}$ R. One might reasonably ask how the feeble tangential component of radiation-pressure could keep up such a state of motion against the smoothing tendency of viscous forces. Jeans calculated that the braking effect would predominate over laminar viscosity only in the region outside a distance from the centre of one-third of the radius, so that the braking effect really leaves the presumed law $Wr^2 = f(\lambda)$ unaccounted for in the region inside a sphere with a radius $\frac{1}{3}$ R, which by itself is a serious difficulty for the theory. However, on any reasonable hypothesis concerning eddy-viscosity in the stars this difficulty of the theory is increased beyond repair. It would seem to be a conservative estimate to compare the vigorous eddies in the granulation of the solar surface with the slow vortices which retard the motion of the oceans on the earth, and which geophysicists estimate to increase the oceanic viscosity a million times, as mentioned above. But an increase in the viscosity coefficient of the sun only by a factor 1000 moves the critical surface from $r = \frac{1}{3}$ R to somewhere in the solar atmosphere, so that, until proofs of the contrary are given, we would seem to be left with the conclusion that the braking effect of radiation has scarcely any astronomical importance whatever, as far as its power of creating a space variation of angular velocity is concerned.

By the same argument we may conclude that, other agencies being absent, viscosity would be amply sufficient to smooth out any initial peculiarity of the internal motion of the sun, and make it settle down in a state of constant angular velocity. In fact, Jeans * estimated that the laminar viscosity of the sun is sufficiently great to smooth out shearing velocities in the course of 10^{14} years. This is such a long time that one would conclude that viscosity plays no important part in the motion of the sun and the stars. Assuming as before eddy-viscosity to be (at least) a million times more effective than laminar viscosity, this figure is cut down to 100 million years only, which is a rather short time interval measured on the cosmic time scale. When, therefore, the sun shows unmistakable signs of having a variable angular velocity there must be some powerful agencies at work which continually create variations in angular velocity as fast as they are smoothed out

* *Loc. cit.*, p. 265. *Cf.* also T. Wilsing and E. Persico, *loc. cit.*

by eddy-viscosity. Since we have seen that radiation-pressure is ruled out as a possible cause, we must look for some other agency, and then the circulation scheme of Eddington seems to be the one best suited for the purpose.

If the convection currents postulated by Eddington would take place mainly in a radial direction, in the same manner as the flux of radiation, we should obviously be led back to the same law $Wr^2 = f(\lambda)$ as in Jeans' theory. This fact was clearly perceived by Faye * in his famous paper on solar phenomena of 1865, which laid the foundation-stone to the modern view of solar thermodynamics. Faye also knew that it was only necessary to assume the sun to possess a spheroidal core rotating with constant angular velocity in order to explain in this way the equatoreal acceleration of the sun. There is, however, an essential difference between Faye's and Eddington's theories as regards the dynamics of the problem, inasmuch as Faye assumed the convection currents to be moving mainly up and down, and to serve as carriers of energy from the interior to the surface. In Eddington's scheme the transfer of energy is effected by radiation, and the circulating currents serve only to produce a redistribution of temperature along the equi-density surfaces such as to satisfy the equations of motion, which demand a general circulation upwards along the axis of rotation and downwards in the equatoreal plane. This fact might seem to imply an essential departure from Faye's theory also as regards its kinematical aspects, but this need not be the case. In fact, a large scale current of the kind imagined by Eddington can hardly be stable. By all analogies from geophysics one would expect the circulatory motion of the star to break up into smaller circulating currents working together like toothed wheels, this being mainly instigated by the force of Corioli. The triple structure of the terrestrial atmosphere (on each hemisphere) shows this tendency clearly, and still more instructive examples are probably to be viewed in the belted structure of the atmospheres of Jupiter and Saturn. If the sun's interior is really broken up into such belted currents we may not be far off from the conditions contemplated by Faye, and an equatoreal acceleration of the sun of the observed kind might ensue. But it must probably be admitted that a rigorous theory of the observed law of solar rotation will require a much more thorough analysis than the one suggested above.

* Hervé Faye, *Comptes Rendus d. l'Acad. des Sc.*, **60**, 89, 138, 1865.

Oslo University Observatory, Norway:
1928 *September* 29.

The Electrical Conductivity of Stellar Matter.

By S. Chapman, M.A., D.Sc., F.R.S.

1. The electrical conductivity of stellar matter is of interest in connection with the theory of the sun's magnetic field. Except possibly for the densest parts of dwarf stars it seems legitimate to calculate it by means of the ordinary kinetic theory of gases. The evaluation of certain integrals involved in the usual formulæ, which become infinite when the law of interaction between the particles is that of the inverse square, will be made as in my former paper,* "Diffusion and Viscosity in Giant Stars." E. Persico's more refined treatment in a recent paper,† taking account of the induced charge surrounding each particle, by a modification of the method of Debye and Hückel ‡ suggested by A. S. Eddington,§ has shown that no serious error is involved in the simpler method.

2. It will, for simplicity, be supposed that the stellar matter near any point consists of free electrons together with positive atomic ions of one kind only; the suffix $_1$ will refer to the electrons, and $_2$ to the ions. The masses will be denoted by m_1, m_2, and the charges by e_1, e_2, where $e_1 = -e$, e being the elementary charge $1{\cdot}59 \cdot 10^{-20}$ e.m.u., and $e_2 = Ze$. The numbers per c.c. being n_1, n_2, we have

$$(1) \qquad n_1e_1 + n_2e_2 = 0,$$

or

$$(2) \qquad Z = n_1/n_2,$$

since the matter is, to a high degree of approximation, electrically neutral.

Let c_1, c_2 denote the mean velocities of the two sets of particles; it will be supposed, for simplicity, that they differ solely owing to unequal accelerations of the particles arising from forces P_1, P_2 to which they are subject, and not through any other cause such as non-uniformity of composition, pressure, or temperature. Then ||

$$(3) \qquad c_1 - c_2 = \frac{(n_1 + n_2)(m_2P_1 - m_1P_2)}{n_1m_1 + n_2m_2} \frac{D_{12}}{kT},$$

where D_{12} denotes the coefficient of diffusion, T the absolute temperature, and $k = 1{\cdot}37 \cdot 10^{-16}$.

These motions correspond to an electric current $n_1e_1c_1 + n_2e_2c_2$ or, by (1), $n_1e_1(c_1 - c_2)$. If the forces are due to an electric field of intensity E, then $P_1 = Ee_1$, $P_2 = Ee_2$, and the electrical conductivity σ is obtained by dividing the expression for the current by E. Thus

$$(4) \qquad \sigma = \frac{(n_1 + n_2)(m_2e_1 - m_1e_2)}{n_1m_1 + n_2m_2} \frac{D_{12}}{kT},$$

* *M.N.*, **82**, 292, 1922.
† E. Persico, *M.N.*, **86**, 93, 1926.
‡ *Phys. Zeit.*, **24**, 185, 1923.
§ A. S. Eddington, *M.N.*, **86**, 1, 1925.
|| *Phil. Trans.*, A, **217**, 115, 1917.

which gives σ in electromagnetic units if e_1, e_2 are expressed in these units.

The coefficient of diffusion for a gas of the kind here considered is given to a first (fairly accurate) approximation by the following formula quoted (with slight changes of notation) from my former paper * :—

$$(5) \quad . \quad . \quad D_{12} = \frac{3}{n_1 + n_2}\left(\frac{\pi}{2}\frac{m_1 + m_2}{m_1 m_2}\right)^{\frac{1}{2}}\frac{(kT)^{\frac{5}{2}}}{I_1 c^4 {e_1}^2 {e_2}^2},$$

where I_1 is a pure number (*cf.* § 3), and the factor c^4, where c denotes the velocity of light, is inserted because the electromagnetic instead of the electrostatic unit of charge is here used. On substituting from (5) into (4), and ignoring the small quantities m_1, $n_1 m_1$, $m_1 e_2$ when combined with the much larger quantities m_2, $n_2 m_2$, $m_2 e_1$ (respectively), and also making use of (1), (2), we find that

$$(6) \quad . \quad . \quad . \quad \sigma = 3\left(\frac{\pi}{2m_1}\right)^{\frac{1}{2}}\frac{(kT)^{\frac{3}{2}}}{I_1 Z c^4 e^2}.$$

This very simple formula indicates that σ depends on the universal constants m_1, e, k, c together with I_1, T, Z, these three alone varying with depth in the star.

3. The integral I_1 is given approximately by

$$(7) \quad . \quad . \quad . \quad I_1 = 2\pi \log_e (1 + a^2),$$

where

$$(8) \quad . \quad . \quad a = 3kTd/c^2 ee_2, \qquad d^3 = 1/(n_1 + n_2);$$

if ρ denotes the density $n_1 m_1 + n_2 m_2$, or, very approximately, $n_2 m_2$, we have $n_1 + n_2 = (1 + Z)n_2 = (1 + Z)\rho/m_2$, so that

$$(9) \quad . \quad . \quad . \quad a = \frac{3k{m_2}^{\frac{1}{3}}}{c^2 e^2 Z(1 + Z)^{\frac{1}{3}}}\frac{T}{\rho^{\frac{1}{3}}}.$$

Taking $T^3\rho$ to be constant at all depths in a star,† and assuming m_2 to be constant, the value of a can be expressed in terms of Z and the values a_c, Z_c at the centre of the star as follows :—

$$(10) \quad . \quad . \quad . \quad a = \frac{Z_c}{Z}\left(\frac{1 + Z_c}{1 + Z}\right)^{\frac{1}{3}} a_c;$$

thus if $Z_c = 25$, and if Z decreases to 1 near the outside of the star, a increases in the ratio $25(13)^{\frac{1}{3}}$ or 59, while the change in I_1 is approximately an *addition* of $2\pi \log_e 59^2$ or 51·3 in passing from the centre to the outside.

For a giant star in which $\rho_c = 0{\cdot}1$, $T_c = 7 \cdot 10^6$, $Z = 25$, $m_2 = 50 m_H$ (where m_H denotes the mass of the hydrogen atom), $a_c = 16$, and $(I_1)_c = 35$; it increases to 86 on passing outwards.

For the sun, in which‡ $\rho_c = 76$, $T_c = 4 \cdot 10^7$, and assuming that

* *M.N.*, **82**, 292, 1922.
† A. S. Eddington, *The Internal Constitution of the Stars*, p. 120, 1926.
‡ *Ibid.*, p. 149.

$Z = 25$, $m_2 = 50m_H$, as in the preceding example, $a_c = 10$, $(I_1)_c = 29$, while at the outside $I_1 = 80$.

4. The ratio of σ at any point in a star to its central value σ_c is given by

$$\frac{\sigma}{\sigma_c} = \frac{(I_1)_c}{I_1}\frac{Z_c}{Z}\left(\frac{T}{T_c}\right)^{\frac{3}{2}}; \qquad (11)$$

of the three factors on the right, the last is the most important. Corresponding to the two cases considered in § 3, it is found that

$$\sigma_c = 2{\cdot}1 \,.\, 10^{-5} \text{ (giant star)}, \qquad \sigma_c = 3{\cdot}4 \,.\, 10^{-4} \text{ (sun)}. \qquad (12)$$

At a level where $T = 10{,}000°$, and assuming that the corresponding value of Z is approximately 1, so that the "outside" values of I_1 may be used, the values of σ are found from (11) to be :—

$$\sigma = 1{\cdot}1 \,.\, 10^{-8} \text{ (giant star)}, \quad \sigma = 1{\cdot}2 \,.\, 10^{-8} \text{ (sun)}; \qquad (13)$$
$$T = 10{,}000, \; Z = 1.$$

The corresponding densities, assuming $T^3\rho$ constant, are $3 \,.\, 10^{-10}$, $1{\cdot}2 \,.\, 10^{-9}$.

Whether or not the estimates of σ_c are affected by the breakdown of the classical statistical mechanics, the values for the less dense regions seem likely to be reasonably accurate. As applied to a region where T and Z are known, (6) is nearly independent of the details of the theory of stellar constitution, which enter only slightly through the evaluation of I_1.

5. As might be expected, in view of the highly ionised condition of stellar matter, the above values of σ_c are high, though their order of magnitude does not transcend that for metals like copper and silver at ordinary temperatures (for which $\sigma = 6 \,.\, 10^{-4}$ e.m.u.) ; at very low temperatures metals become far more conducting, for reasons only partly understood as yet.

6. In the outer layers of a star σ is much smaller, being of the order 10^{-8} in the region where $T = 10{,}000°$; nevertheless this may be regarded as considerable, and likely to be important in connection with the sun's magnetic field. It is of interest to compare it with the approximate values of σ for various terrestrial materials, as follows :—

Sun or star, where $T = 10{,}000°$	$\sigma = 10^{-8}$ e.m.u.
Sea water	$5 \,.\, 10^{-11}$
Dry earth or rock	10^{-15} to 10^{-16}
Earth's core *	$4 \,.\, 10^{-13}$
Heaviside layer,† at 140 km. height	10^{-10}.

* *Phil. Trans.*, A, **218**, 41, 1919.

† P. O. Pedersen, *The Propagation of Radio Waves*, Copenhagen, 149, 1927 ; the value quoted is the estimated maximum for the conductivity in the direction of the magnetic field. It is probably uncertain to the extent of a factor 10.

7. In the outermost layers of the sun (upwards from about 100 km. below the photosphere) the conductivity in the direction *transverse* to the magnetic field is reduced below the value given by (6), which still applies, however, to the conductivity along the field. In this region the density is so low that the electrons can freely spiral round the lines of magnetic force between collisions, and gain little or no mean motion in the direction of an applied electric field. When they can execute three or more revolutions in their spiral paths between collisions, their contribution to σ transverse to the field is reduced below $\frac{1}{300}$ of the value (6), and then the contribution of the ions to σ, which is of order $(m_1/m_2)^{\frac{1}{2}}$ times (6)—$(m_1/m_2)^{\frac{1}{2}}$ being about $\frac{1}{300}$—becomes significant.

On the Radial Limitation of the Sun's Magnetic Field.

By S. Chapman, M.A., D.Sc., F.R.S.

§ 1. The principal properties of the sun's general magnetic field, discovered by Dr. Hale and further observed solely at Mount Wilson by him and his colleagues,* are as follows :—

(*a*) The distribution of the component of magnetic force along the line of sight (that is, in the mean, parallel to the magnetic equatorial plane) is generally similar to that for a uniformly magnetised sphere, being zero at the equator and increasing with latitude.

(*b*) But there are distinct deviations of the distribution of intensity with respect to latitude, from that characterising a uniformly magnetised sphere. Messrs. Seares, van Maanen, and Ellerman have made separate reductions of the observations derived from the latitude zones − 10° to + 10°, and ± (10° to 45°), using the formulæ appropriate to a uniformly magnetised sphere ; the estimates of the polar intensity thus derived are very different, that from the equatorial zone being 1·84 times as great as that for the higher zones.

(*c*) The magnetic intensity decreases very rapidly with increasing radial distance from the sun's centre ; at the lowest level of observation, above the photosphere, the estimated polar intensity is about 50Γ ; it rapidly falls to 10Γ at higher levels ; measurement of smaller field-strengths is not yet practicable. Messrs. Hale, Seares, van Maanen, and Ellerman give the corresponding heights above the photosphere as about 240 and 540 km., indicating that the decrease occurs in 300 km. As these distances did not seem to accord with the conclusions regarding the thickness of the reversing layer derived by Professor E. A. Milne from his theoretical studies, I inquired of him to what levels he would assign the spectral lines by which the magnetic field is measured ; his reply, which I quote with his permission, was that the lines probably originate low in the reversing layer at a level where the partial pressure of free electrons is about 10^{-4} atmos. (or 100

* *Mount Wilson Contributions*, Nos. 71, 72, 148 ; *Mount Wilson Communications*, Nos. 50, 61.

dynes/cm.^{2}), and that the decrease of the field to (practically) zero may not improbably occur in a layer at most 50 km. thick.

(*d*) The magnetic axis nearly coincides with the axis of rotation, and the direction of magnetisation is related to that of rotation in the same way as for the earth. The obliquity of the magnetic axis is about 4°, and the magnetic field rotates about the sun's axis with a synodic period of 31·8 days.

§ 1.1. As in the case of the earth's field, so also for the sun no detailed theoretical explanation of the general magnetic field, or of its principal properties, has yet been found ; but some important considerations on the problem have been given in papers by Sir Joseph Larmor * and Professor Rosseland.†

A possible method of explaining the radial limitation of the sun's

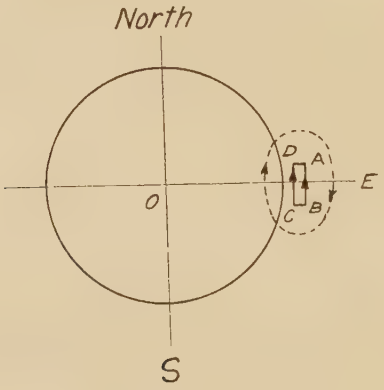


FIG. 1.

field is here proposed, and subjected to preliminary examination. It is shown that a magnetic field proceeding from within the sun must necessarily generate electric currents in the sun's ionised atmosphere, of a kind limiting the outward extension of the field. It is hoped to deal later in more detail with several points that arise, and to work out the general properties of a rare ionised atmosphere in the presence of a magnetic field.

§ 2. The rapid radial outward decrease of the field, which at the equator is horizontal, indicates that the sun's atmosphere is the seat of eastward electric currents. Consider the line-integral of the magnetic intensity H (a vector) round a small circuit ABCD in a meridian plane, as shown in fig. 1 ; AB, CD being circular arcs with centre at O, and AD, BC radii from O. Along AD, BC the component magnetic intensity is zero (or very small), while along AB, CD it has the directions shown, being much greater along CD than along AB ; hence the line-integral

* Sir J. Larmor, *Brit. Assoc. Report*, 1919, p. 159.
† S. Rosseland, *Mount Wilson Contributions*, No. 302.

round ABCD, taken in this sense, is positive, indicating that the circuit is threaded by an electric current flowing downwards into the diagram, that is, eastward on the sun. Such a current would produce a magnetic field having lines of force of the kind illustrated by a dotted line : in the equatorial plane the effect will be to make the northward force decrease outward and increase inward, as observed.

If we ignore any small magnetic susceptibility of the solar material, $\mathrm{div}\,H = 0$, that is, the magnetic field can be divided up by closed re-entrant tubes, directed along H at each point, and such that their cross-section is inversely proportional to H. Such tubes will cross the magnetic equatorial plane at right angles. In other latitudes, since the field practically does not extend beyond the reversing layer, the inclination of the tubes to the horizontal must be small (except near the poles) ; otherwise they would project too far outward.

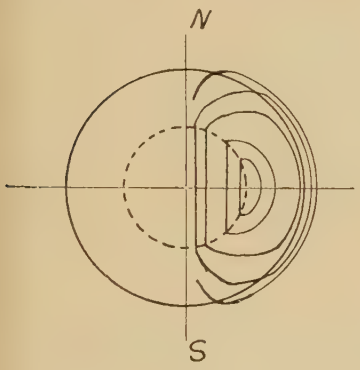


FIG. 2A.

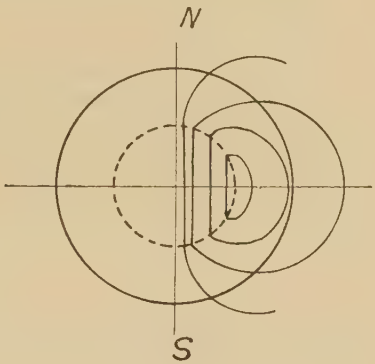


FIG. 2B.

If we imagine the arc AB to generate a cylinder ABB'A' by rotation about NS, and if AB is taken at a radius beyond which the field is practically zero, all the tubes crossing the equatorial plane must do so within this cylinder. The northward tubes observed in the sun's atmosphere must cross the equatorial plane in the southward direction, within the sun. Near the surface H is northward and increasing in intensity as we go downwards ; since at some smaller radius it is reversed in sign (or direction), it must first attain a maximum northward intensity and then decrease again, to zero, afterwards being of the opposite sign. We cannot observe the depth at which the reversal occurs, or the maximum northward intensity ; there may even be more than one reversal of direction between the surface and centre, though this is perhaps unlikely. The simplest picture of the distribution of the sun's field is as represented in fig. 2A, though the relative dimensions of different parts are wholly conjectural, and are chosen only with a view to convenient illustration ; for comparison the distribution of the lines of force is shown in fig. 2B for a uniformly magnetised sphere. These spread far outward into space, whereas in the case of the sun the

tubes of force, shown as straight and southward within the interior, instead of completing their northward course outside the sun, are drawn down into the atmosphere. It is somewhat as if, round an inner magnetised sphere, a highly permeable magnetic shell was placed, in which the tubes of force concentrated their return paths, the space outside being almost completely shielded from the field. In the case of the sun the shielding is not effected by a highly permeable shell, but by a system of eastward electric currents in the atmosphere.

As we proceed inwards from a point E on the equator, at radius R such that the magnetic intensity is there practically zero, the variation of H, reckoned positive if northward, is probably somewhat as

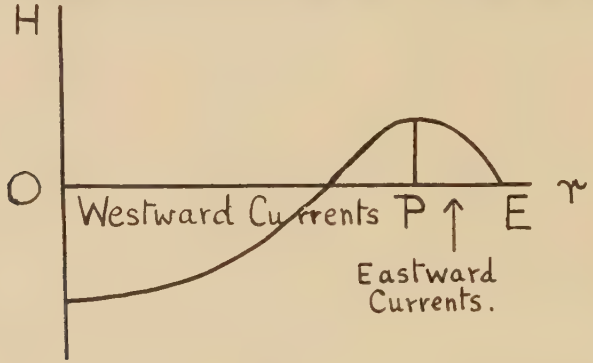


FIG. 3.

shown in fig. 3. The condition that the net number of tubes crossing the equatorial plane is zero may be expressed in the form

$$(1) \quad \ldots \quad \int_0^R r\mathrm{H}dr = 0.$$

By considering a line-integral round a circuit such as ABCD (fig. 1), it appears that, so long as H is varying rapidly with r, the eastward equatorial electric current-intensity i, per unit area in the meridian plane, is given by

$$(2) \quad \ldots \quad i = -\frac{1}{4\pi}\frac{d\mathrm{H}}{dr};$$

but when H is not varying rapidly, the contributions to the line-integral along the sides AD and BC must be taken into account, and (2) must be modified. Between E and (approximately) P the current is eastward, and below P westward, being reversed before the reversal of H.

The main object of this paper is to show how the field proceeding from within, and due to westward electric currents, will set up eastward electric currents in the solar atmosphere. To do this it is necessary to describe certain properties of an ionised gas in the presence of a magnetic field.

§ 3. A particle of mass m and charge e (in electromagnetic units)

moving freely with velocity v in a magnetic field of intensity H, the component velocity transverse to H being u, will describe a spiral path lying on a cylinder having its axis along the field. The radius R of the cylinder (or spiral) is given by

$$(3) \quad . \quad . \quad . \quad . \quad R = mu/eH ;$$

this will be called the spiral-radius. The angular velocity ω with which the spiral is described is given by

$$(4) \quad . \quad . \quad . \quad . \quad \omega = eH/m ;$$

this will be called the spiral-frequency of the particle.

The magnitudes of R and ω are very different for electrons and ions, on account of their widely different masses. It is convenient to indicate, when necessary, to which kind of particle these or other such quantities refer, by attaching the suffix e or i to the corresponding symbol.

§ 4. If a force F of any kind acts on the charge, the motion is governed by the following equations, which for convenience are referred to right-handed axes chosen so that Oz is along H, and Oy normal both to F and H ; the components of F may then be denoted by (X, o, Z), X being that which is transverse to the field :

$$(5) \quad . \quad . \quad . \quad . \quad m\ddot{x} = X + eH\dot{y},$$

$$(6) \quad . \quad . \quad . \quad . \quad m\ddot{y} = -eH\dot{x},$$

$$(7) \quad . \quad . \quad . \quad . \quad m = Z.$$

The motion along the field is clearly independent of the field. The transverse motion is determined by the first integrals of (5) and (6), namely, in terms of the initial velocities (indicated by the suffix o) :

$$(8) \quad . \quad \dot{x} = \dot{x}_0 \cos \omega t + \dot{y}_0 \sin \omega t + (X/eH) \sin \omega t,$$

$$(9) \quad . \quad \dot{y} = -(X/eH)(1 - \cos \omega t) - \dot{x}_0 \sin \omega t + \dot{y}_0 \cos \omega t.$$

The mean value of $\dot{x}$ is zero, showing that the transverse force X has no mean accelerative effect on the free charge ; but it superposes on the otherwise existing motion a fluctuating velocity in the y-direction (normal to F and H) with the mean value $-X/eH$. In vector notation this mean velocity is $[F, H]/eH^2$, where the numerator is the vector product of F and H. It is convenient to give the name *drift* to this mean transverse velocity resulting from the joint action of F and the field ; it will be denoted by V.

A mechanical analogy of this drift is well known : a rough sphere placed upon an inclined plane rotating uniformly about some axis normal to itself acquires no mean motion down the plane, but describes a trochoidal path with its mean motion horizontal, transverse to the component of gravity down the plane.*

* Lamb's *Higher Mechanics*, p. 109, Ex. 8.

§ 5. The drift V will have the mean value $-X/eH$ only if the particle is free or nearly so. In an ionised atmosphere this implies that the charge must be able to spiral many times round the lines of force between collisions. An approximate formula taking account of the frequency of collisions can be derived as follows, by assuming that all the particles (of one kind, ions or electrons) have the same velocity v. Then if l is the mean free path, the mean interval τ between collisions is l/v, and ν the collision-frequency, or number of collisions per particle per second, is $1/\tau$ or v/l. The fraction of the intervals between collisions which lie between t and $t+dt$ is $\nu \exp(-\nu t)dt$, so that the mean drift-velocity is given by

$$(10)\quad V = -(X/eH)\int_0^\infty (1-\cos\omega t)\nu\exp(-\nu t)dt = -\frac{X}{eH}\frac{\omega^2}{\nu^2+\omega^2} = -\frac{X}{eH}\frac{l^2}{R^2+l^2},$$

when ν/ω is small, or l/R large, this reduces to $-X/eH$, while it tends to zero with l/R or ω/ν.

§ 6. If the numbers per c.c. of electrons and ions are equal, *i.e.* if $n_e = n_i$ (or if n_e is any known multiple of n_i), the mean free paths l_e and l_i, being inversely proportional to the density of the gas, may be written

$$(11)\quad . \quad . \quad . \quad l_e = a_e/n_i, \qquad l_i = a_i/n_i,$$

where a_e, a_i are certain constants which (at least to a first approximation) are independent of the temperature,† but depend on the size or fields of force of the ions and electrons. The values of a_e and a_i have not been adequately investigated for a gas in which all the particles are charged, especially in the presence of a magnetic field: I hope later to consider this point in detail, but for the present it is sufficient to adopt rough values which at least are not likely to be of the wrong order of magnitude. It will be assumed that l_i is given by $1/\sqrt{2}\pi n_i d^2$, d being the diameter of an ion, regarded as a rigid elastic sphere; taking $d = 3 \cdot 10^{-8}$ cm.,

$$(12)\quad . \quad . \quad l_i = 2{\cdot}5 \cdot 10^{14}/n_i, \quad \text{or} \quad n_i = 2{\cdot}5 \cdot 10^{14}/l_i.$$

For the electrons we take

$$(13)\quad l_e = 4\sqrt{2}l_i = 1{\cdot}4 \cdot 10^{15}/n_i, \quad \text{or} \quad n_i = 1{\cdot}4 \cdot 10^{15}/l_e,$$

according to the approximate theory given by Loeb (*Kinetic Theory of Gases*, p. 49).

In the lower levels of the solar atmosphere, taken to be at 5500° K, and in thermodynamic equilibrium,

$$(14)\quad . \quad . \quad . \quad \tfrac{1}{2}m_i v_i^2 = \tfrac{1}{2}m_e v_e^2 = \tfrac{3}{2}k\mathrm{T},$$

where $k = 1{\cdot}37 \cdot 10^{-16}$; for the transverse component velocity u, $u^2 = \frac{2}{3}v^2$. Hence, taking the ions to be Ca^+ ions, and inserting the

* In this last form of V the difference between u and v is neglected.

† Except in so far as this affects the ratio n_e/n_i.

values $m_i = 6{\cdot}56 \cdot 10^{-23}$ gm., $m_e = 8{\cdot}9 \cdot 10^{-28}$ gm., $e = 1{\cdot}59 \cdot 10^{-20}$ e.m.u., we find $u_i = 1{\cdot}5 \cdot 10^5$ cm./sec., $u_e = 4{\cdot}1 \cdot 10^7$ cm./sec., and $m_i u_i/e = 6{\cdot}2 \cdot 10^2$, $m_e u_e/e = 2{\cdot}3$, so that

$$(15) \quad . \quad . \quad R_i = 6{\cdot}23 \cdot 10^2/H, \qquad R_e = 2{\cdot}3/H,$$

$$(16) \quad . \quad . \quad \omega_i = 2{\cdot}4 \cdot 10^2 H, \qquad \omega_e = 1{\cdot}8 \cdot 10^7 H,$$

$$(17) \quad . \quad . \quad \nu_i = 7{\cdot}4 \cdot 10^{-10} n_i, \qquad \nu_e = 3{\cdot}6 \cdot 10^{-8}\, n_i.$$

Thus where H = 50, R_i is 12·5 cm., and R_e is $4{\cdot}6 \cdot 10^{-2}$ cm. The value of n_e or n_i corresponding to $l_i = 12{\cdot}5$ cm. is $2 \cdot 10^{13}$, while that corresponding to $l_e = 4{\cdot}6 \cdot 10^{-2}$ cm. is $3{\cdot}0 \cdot 10^{16}$; the corresponding electron-pressures, at 5500° K, expressed in atmospheres (= 10^6 dynes/cm.^{2}), are $1{\cdot}5 \cdot 10^{-5}$ and $2{\cdot}3 \cdot 10^{-2}$. According to Professor Milne the electron-pressure p_e in the sun at the depth of complete opacity (the top of the photosphere) is $5 \cdot 10^{-5}$, and if so the pressure in the layer where H = 50 cannot exceed this value. Hence in this region the pressure is far below that for which $R_e = l_e$; thus l_e/R_e is large, or ν_e/ω_e small, and the electrons will take up the full drift-velocity $-X/eH$. For the ions the case is different; where H = 50, R_i/l_i may be as much as 3 (*i.e.* $5 \cdot 10^{-5} \div 1{\cdot}5 \cdot 10^{-5}$), though this is an upper limit of the ratio; the corresponding value of ν_i/ω_i is 4, so that the ions will take up only 1/17 of the drift-velocity. Actually the fraction may be somewhat greater, because part of the systematic drift-motion will be preserved after collisions; the exact mean drift can be determined only from a detailed statistical study of the gas, which I hope to undertake.

§ 7. When the applied force is due to an electric field E, then $X = E_x e$. The (full) drift-velocity is then $-E_x/H$; this is independent of the mass, charge, and sign of the particle. In a neutral ionised medium rare enough for both sets of charges to take up this drift, all alike will have the same velocity, and the transverse component of the electric field will produce a motion of the whole mass, but no electric current, either in its own direction or in the orthogonal transverse direction y; that is, the medium is electrically non-conducting for transverse electric fields, though for electric gradients along the magnetic field the gas is normally conducting.

§ 8. If the applied force is gravitational, $X = mg_x$, and the (full) drift-velocity is $-mg_x/eH$. This is far greater for an ion than for an electron (and of opposite sign if, as in this paper, the ions considered are all positive).

Near the sun's surface, in the equatorial plane, H (and Oz—*cf.* § 4) is horizontal and northward, and gravity g (= $2{\cdot}74 \cdot 10^4$ cm./sec.^{2}) is wholly transverse; taking the Ox axis upward and vertical, the Oy axis is horizontal and eastward. Hence $V_i = 1{\cdot}13 \cdot 10^2/H$ eastward; this is 2·26 cm./sec. if H = 50, though in the sun's atmosphere the ions are not free to take up more than a small fraction of this drift at the level where H = 50. Similarly $V_e = 1{\cdot}54 \cdot 10^{-3}/H$ westward. These velocities are very minute compared with the (random) thermal motions, and apart from the small superposed drift the charges describe spiral paths with radii of practically the same size as if gravity did not operate.

The eastward drift of the positive ions and the westward drift of the electrons correspond to an eastward electric current, of intensity

$$(18) \quad . \quad . \quad i = (n_i V_i - n_e V_e)e = n_i e(V_i - V_e)$$

if, as we assume, $n_i = n_e$.

§ 9. If the drift-currents in the sun's atmosphere were due solely to gravitation, the contribution made by the electrons would be negligible compared with that made by the ions, on account of the far smaller value of V_e than V_i, and in spite of the greater thickness of the atmospheric layer in which the electrons are free to take up the drift-motion, by reason of their smaller value of R/l. There is, however, a vertical electrostatic field set up by the tendency of the light electrons to spread out farther in the vertical direction than the heavier ions; this was first pointed out by Dr. Pannekoek,* and has been further discussed by Professors Rosseland and Milne.† Except for a very rare layer at the extreme limit of the atmosphere (where the pressure is less than 10^{-24} atmos.) the field is such that

$$(19) \quad . \quad . \quad . \quad . \quad eE = \tfrac{1}{2}(m_i - m_e)g;$$

this is upward on the positive ion, and downward on the electron; thus in all,

$$(20) \quad . \quad . \quad . \quad X_i = X_e = \tfrac{1}{2}(m_i + m_e)g_x.$$

Consequently the (full) drift-velocity is the same for an electron as for an ion, and since the electrons are free to take up this velocity at much higher pressures than the ions, the main drift-currents in the sun's atmosphere are carried by the electrons: only in the outer rarer layers, if at all, can the ions contribute, and there each ion carries as much current as an electron. It will be evident from the following discussion that if electrons and ions were of similar (atomic) mass the drift-currents would be far smaller than are actually permitted by the great disparity of mass.

§ 10. The eastward drift-current-intensity i in the sun's atmosphere is obtained by inserting the values of V_i and V_e from (10) into (18), using (20); this gives

$$(21) \quad . \quad . \quad . \quad i = \tfrac{1}{2}Kn_i(m_i + m_e)g_x/H,$$

where

$$(22) \quad K = \frac{\omega_i^2}{\nu_i^2 + \omega_i^2} + \frac{\omega_e^2}{\nu_e^2 + \omega_e^2} = \frac{1}{1 + \beta_e(n_i/H)^2} + \frac{1}{1 + \beta_i(n_i/H)^2}$$

$$= \frac{1}{1 + 4 \cdot 10^{-30}(n_i/H)^2} + \frac{1}{1 + 9{\cdot}5 \cdot 10^{-24}(n_i/H)^2}$$

using (16), (17). The factor K lies between 0 and 2. The mass m_e in (21) can be ignored. The coefficients β are $(mu/e\alpha)^2$, with suffix e or

* A. Pannekoek, *B.A.N.*, **1**, 110, No. 19 (1922); S. Rosseland, *M.N.*, **84**, 720, 1924.

† E. A. Milne, *Proc. Camb. Phil. Soc.*, **22**, 495, 1925.

i throughout, α being given by (11); the numerical values of β here used are subject to revision.

The above equation will be applied to the part of the sun's atmosphere where H varies sufficiently rapidly along the radius for (2) to be valid, *i.e.*

$$(23) \quad . \quad . \quad . \quad d\mathrm{H}/dr = -2\pi \mathrm{K} m_i g_x n_i/\mathrm{H},$$

or

$$(24) \quad . \quad . \quad . \quad d\mathrm{H}^2/dr = -4\pi \mathrm{K} m_i g_x n_i.$$

Since (§ 2) H is almost horizontal throughout this layer (except quite close to the poles, *cf.* § 18), gravity is almost wholly transverse to H, and we may write $g_x = g$.

§ 11. In the lower reversing layer the temperature is nearly constant * and the atmosphere is supported mainly by gas-pressure, so that approximately

$$(25) \quad . \quad . \quad n_i = n_0 \exp\{-m_i g(r - r_0)/2k\mathrm{T}\},$$

where n_0 is the value of n_i at r_0; r_0 may conveniently be taken to refer to the level (at the equator) where H = 50. On substituting from (25) into (24), the equation can be integrated if we suppose g, T, and K to be all constant (this assumption is probably sufficiently accurate as regards g and T; its validity as regards K is examined in § 13); the result is

$$(26) \quad \mathrm{H}_0^2 - \mathrm{H}^2 = 8\pi \mathrm{K} k n_0 \mathrm{T}[1 - \exp\{-m_i g(r - r_0)/2k\mathrm{T}\}],$$

where H_0 denotes the value of H at r_0. Since the ion- or electron-pressures p_i, p_0 at r and r_0 are given by

$$(27) \quad . \quad . \quad . \quad p_i = k n_i \mathrm{T}, \qquad p_0 = k n_0 \mathrm{T},$$

(26) can be written in the form

$$(28) \quad . \quad . \quad \mathrm{H}^2 = \mathrm{H}_0^2 - 8\pi \mathrm{K}(p_0 - p_i)$$
$$= \mathrm{H}_0^2 - 8\pi \mathrm{K} p_0\{1 - \exp(-x/h)\},$$

if we write

$$(29) \quad . \quad . \quad . \quad . \quad x = r - r_0,$$

$$(30) \quad . \quad h = 2k\mathrm{T}/m_i g = 8{\cdot}4 \,.\, 10^5 \text{ cm.} = 8{\cdot}4 \text{ km.},$$

taking T = 5500° K; h is the height of the "homogeneous atmosphere," or the vertical thickness in which p_i or n_i decrease in the ratio 1 : 2·718.

From (28) it appears that so long as K is constant H^2 is a linear function of the pressure in this isothermal layer wherein p_i decreases exponentially. Since p_i cannot be less than zero, the maximum diminution of H^2 in this layer, below the value H_0, is $8\pi \mathrm{K} p_0$, *i.e.*

$$(31) \quad . \quad . \quad . \quad \mathrm{H}_0^2 - \mathrm{H}_\infty^2 = 8\pi \mathrm{K} p_0.$$

Moreover the reduction in H^2, whatever its amount, occurs very

* A. S. Eddington, *The Internal Constitution of the Stars*, p. 362.

rapidly, nine-tenths being completed when $\exp(-x/h) = 1/10$, or $x = 2{\cdot}3\,h = 19$ km.; at this level $p_i = \frac{1}{10}p_0$.

§ 12. In the sun's atmosphere a reduction of H from 50 to 10 gauss is actually observed, and the diminution may be to still lower values which, however, are not measurable at present. If the reduction is due to the cause here suggested, and we take the minimum value of H as 10 (corresponding to H_∞), then $8\pi K p_0 = 2500 - 100 = 2400$; if the minimum value is actually zero,* $8\pi K p_0 > 2500$. Assuming $K = 1$, corresponding to freedom for the electrons, but not for the ions, to take up the drift-velocity, then p_0 is $2500/8\pi$ or $2400/8\pi$, *i.e.* 95 or 99 dynes/cm²., or 10^{-4} atmos. This is twice as great as Professor Milne's estimate of p_i at the base of the reversing layer, but the agreement between two estimates of the pressure, derived in such totally different ways, is much more remarkable than the discrepancy. The latter would be removed if the value of H at the base of the layer were 35 gauss instead of 50, but if the measurements of H do not permit this supposition, there seems little room for the present estimate of p_0 to be in error. The only point that requires examination is whether K is nearly unity, as supposed, at least throughout the layer of 19 km. in which H^2 suffers $\frac{9}{10}$ of its full reduction, and p_i falls to $\frac{1}{10}p_0$. In this layer n_i/H steadily decreases, and its maximum value is at $x = 0$, being n_0/H_0, or $1{\cdot}3 \,.\, 10^{14}/50 = 2{\cdot}6 \,.\, 10^{12}$ (taking $p_0 = 10^{-4}$ atmos.); hence the first term of K—*cf.* (22)—is practically unity at and above r_0. The maximum value of the second term occurs at the top of the layer, where $n_i/H = 1{\cdot}3 \,.\, 10^{13}/14$, taking $H_\infty = 10$; hence the second term will not exceed about $\frac{1}{10}$ in this layer. If $H_\infty = 10$, K rises to 2 shortly above $x = 19$ km., because H varies only slightly above this region, while n_i is rapidly reduced.†

The thickness of the layer, about 20 kms., in which H is reduced from 50 to about 10 gauss (depending on the value of H_∞, which in turn depends on p_0) is in good agreement with Professor Milne's estimate of the difference of level of origin of the spectral lines associated with $H = 50$ and $H = 10$ respectively (§ 1).

These results seem to establish a presumption in favour of the above theory of the radial limitation of the sun's field. The above completes the main argument of the paper, but there are numerous other points that deserve consideration, though the conclusions reached are not so readily checked by independent evidence.

§ 13. The exact mode of decrease of H in the reversing layer, taking account of the variation of K, is best determined by numerical calculation. For the present the atmosphere will be supposed isothermal, so that (25) is valid. The ratio n_i/H decreases upwards at first, so that K slowly increases towards the value 2. Over the first part of the range it is convenient to transform (24), by introducing the variables

$$(32) \quad . \quad . \quad y = H^2/H_0^2, \qquad \xi = \exp(-x/h);$$

* In the succeeding paper evidence is brought forward indicating that the field is reduced to zero below the chromosphere.

† It seems most likely, however, that K never much exceeds unity, and that $H_\infty = 0$.

then

$$(33)\quad K = \frac{1}{1 + 4.10^{-30}(n_0/H_0)^2\xi^2/y} + \frac{1}{1 + 9{\cdot}5\,.\,10^{-24}(n_0/H_0)^2\xi^2/y},$$

and

$$(34)\quad \frac{dy}{d\xi} = \frac{8\pi p_0}{H_0^2}K.$$

The values of y and ξ at r_0 (or $x = 0$) are each unity; also $H_0 = 50$. It may be noted that ξ, being equal to p/p_0, is a measure of the pressure at any level.

If y/ξ^2 becomes very small, it is possible to write $K = Ay/\xi^2$, where $1/A = 4.10^{-30}(n_0/H_0)^2$; it is then convenient to rewrite (34) in the form

$$(35)\quad \frac{d\log y}{d(1/\xi)} = -\frac{8\pi p_0}{H_0^2}A \equiv B,$$

or

$$(36)\quad \log y = -(B/\xi) + \text{constant}.$$

The final decrease of y or H according to (36) is very rapid.

For intermediate values of y, it is convenient to use the variable $u \equiv y/\xi$ instead of y, and to take x as the independent variable; the corresponding equation is

$$(37)\quad \frac{du}{dx} = \frac{u - K}{h}.$$

The following tables show how y or H decrease with the pressure, assuming different values of p_0 at the level where $H = H_0 = 50$. These and the later tables have been calculated by Miss M. C. Gray.

TABLE I.

$8\pi p_0 = H_0^2$.

x (km.)	0	10·1	19·3	25·1	27·0	27·8
p/p_0	1	·3	·1	$5{\cdot}01.10^{-2}$	$4{\cdot}0.10^{-2}$	$3{\cdot}6.10^{-2}$
K	1·015	1·045	1·096	1·084	1·040	1·009
y	1	·283	$7{\cdot}03.10^{-2}$	$1{\cdot}54.10^{-2}$	$4{\cdot}4.10^{-3}$	$8{\cdot}3.10^{-4}$
H	50	26·6	13·2	6·2	3·3	1·4

In this case, between $x = 27{\cdot}9$ and 28·1 km., H decreases from about 1 gauss to less than 10^{-5} gauss. At this point p is about $3{\cdot}5\,.\,10^{-6}$ atmos., corresponding to a region well below the chromosphere.

TABLE II.

$8\pi p_0 = \frac{3}{4}H_0^2$.

x (km.)	0	10·1	19·3	27·0	32·8	∞
p/p_0	1	·3	·1	·04	·02	0
K	1·026	1·117	1·416	1·766	1·918	2
y	1	·448	·266	·195	·168	·138
H	50	33·5	25·8	22·1	20·5	18·6

TABLE III.

$8\pi p_0 = \frac{1}{2}H_0^2$.

x	0	10·1	19·3	32·8	∞
p/p_0	1	·3	·1	·02	0
K	1·057	1·289	1·735	1·983	2
y	1	·607	·461	·386	·366
H	50	38·9	34·0	31·0	30·2

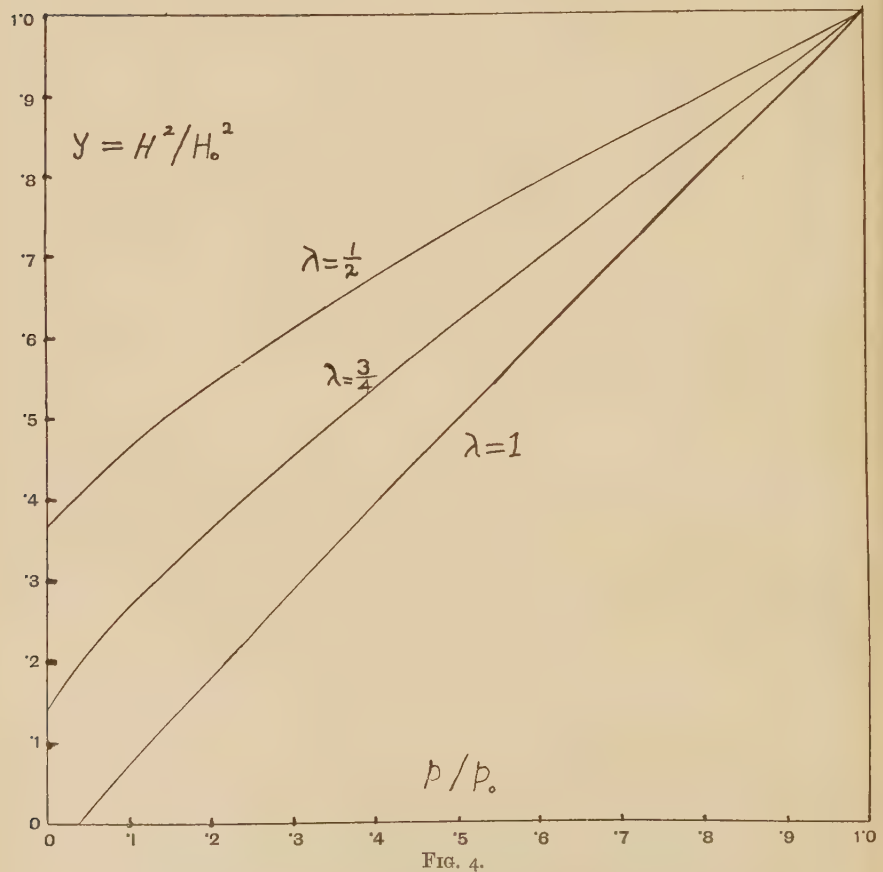


FIG. 4.

In Tables II. and III., H attains a limiting value exceeding 10 gauss; since H is observed to decrease to 10 gauss, it is clear that at the level where H = 50 the value of p_0 must definitely exceed $\frac{3}{4}H_0^2/8\pi$, or $0{\cdot}75 \cdot 10^{-4}$ atmos. The minimum value of p_0 consistent with the observed reduction of H is approximately $0{\cdot}9 \cdot 10^{-4}$ atmos., agreeing with the rough calculation in § 12.

The results in Tables I.–III. are illustrated in fig. 4, which shows the relation between H^2/H_0^2, or y, and p/p_0; the scale of x is also indicated,

but whereas this depends on the assumed temperature, the relation between y and p/p_0 depends only on the temperature remaining constant. The curves would be only slightly altered if the actual downward increase of temperature were allowed for.

§ 14. The variation of H as we proceed in the opposite direction will next be considered, *i.e.* as we descend into the photosphere. Calculations have been made by the equations given in § 13, assuming a constant temperature, and considering alternative values of $\lambda = 8\pi p_0/H_0^2$, namely, $\frac{1}{2}$, 1, 2, 10. It has been shown that λ must exceed the first of these values.

As we descend, K decreases, at first slowly, but finally very quickly, and a limiting value of H is reached, at depths which in the above four cases are approximately 130, 125, 115, 105 km. The limiting values of H are practically equal, about 11,300 gauss; actually the limiting value of y and H is slightly reduced by increasing λ. This is easily seen by ignoring the second term in K, which is small for the values of λ here considered, and writing $\lambda\xi = q$. Then (34) may be written in the form

$$\frac{dy}{dq} = \frac{1}{1 + aq^2/y},$$

the initial values being $q = \lambda$, $y = 1$; alternatively, if we compare values of y for the same value of q, say $q = 10$, $y = 1/\lambda$ approximately. The value of dy/dx for any later value of q is clearly greater, the smaller the value of λ, or the greater the value of y; hence the limiting value of y decreases as λ increases. But as the limiting value of y is about $5 \cdot 10^4$ the relatively small difference between the initial values of y (for $\lambda = \frac{1}{2}$ to $\lambda = 10$) scarcely influences the final value, and beyond $q = 1000$ the value of y is almost independent of the initial value and depends practically solely on q. The limiting value depends materially on the assumed value of α_e (§ 6).

The values of K, y, H at different depths $(-x)$ below r_0 are shown in the following tables, for the above four values of λ. The values of y are illustrated also in fig. 5.

Depth $(-x)$.		0.	19·3.	38·6.	57·9.	77·2.	87·0.	96·5.	106·0.	115·8.	125·4.	135·1.
p/p_0.		1.	10.	100.	1000.	10^4.	$10^{4\cdot5}$.	10^5.	$10^{5\cdot5}$.	10^6.	$10^{6\cdot5}$.	10^7.
λ	K	1·057	1·003	·999	·986	·871	·650	·287	·057	·0070	$7{\cdot}4.10^{-4}$	8.10^{-5}
=	y	1	5·55	50·5	497	$4{\cdot}67.10^3$	$1{\cdot}28.10^4$	$2{\cdot}78.10^4$	$4{\cdot}18.10^4$	$4{\cdot}89.10^4$	$5{\cdot}13.10^4$	$5{\cdot}19.10^4$
$\frac{1}{2}$	H	50	118	355	1113	$3{\cdot}41.10^3$	$5{\cdot}65.10^3$	$8{\cdot}33.10^3$	$1{\cdot}02.10^4$	$1{\cdot}11.10^4$	$1{\cdot}13.10^4$	$1{\cdot}14.10^4$
λ	K	1·015	1·001	·997	·973	·758	·431	·116	·016	·0018	2.10^{-4}	
=	y	1	10·0	99·9	986	$8{\cdot}8.10^3$	$2{\cdot}1.10^4$	$3{\cdot}7.10^4$	$4{\cdot}6.10^4$	$4{\cdot}97.10^4$	$5{\cdot}09.10^4$	
1	H	50	158	499	1570	$4{\cdot}7.10^3$	$7{\cdot}3.10^3$	$9{\cdot}6.10^3$	$1{\cdot}08.10^4$	$1{\cdot}11.10^4$	$1{\cdot}13.10^4$	
λ	K	1·004	1·000	·994	·946	·581	·218	·0378	·0044	5.10^{-4}	5.10^{-5}	
=	y	1	19·0	198·5	1945	$1{\cdot}54.10^4$	$3{\cdot}10.10^4$	$4{\cdot}36.10^4$	$4{\cdot}91.10^4$	$5{\cdot}12.10^4$	$5{\cdot}18.10^4$	
2	H	50	218	704	$2{\cdot}20.10^3$	$6{\cdot}20.10^3$	$8{\cdot}80.10^3$	$1{\cdot}04.10^4$	$1{\cdot}11.10^4$	$1{\cdot}13.10^4$	$1{\cdot}135.10^4$	
λ	K	·997	·997	·972	·759	·118	·0165	·0018	2.10^{-4}	$<10^{-5}$		
=	y	1	89·8	976	8724	$3{\cdot}69.10^4$	$4{\cdot}65.10^4$	$5{\cdot}01.10^4$	$5{\cdot}12.10^4$	$5{\cdot}2.10^4$		
3	H	50	474	1560	$4{\cdot}66.10^3$	$9{\cdot}59.10^3$	$1{\cdot}08.10^4$	$1{\cdot}12.10^4$	$1{\cdot}13.10^4$	$1{\cdot}14.10^4$		

Although the estimated values of H_b, the magnetic intensity at the base of the drift-current layer, represent a great extrapolation from the observed values of H, it seems certain that the increase must occur

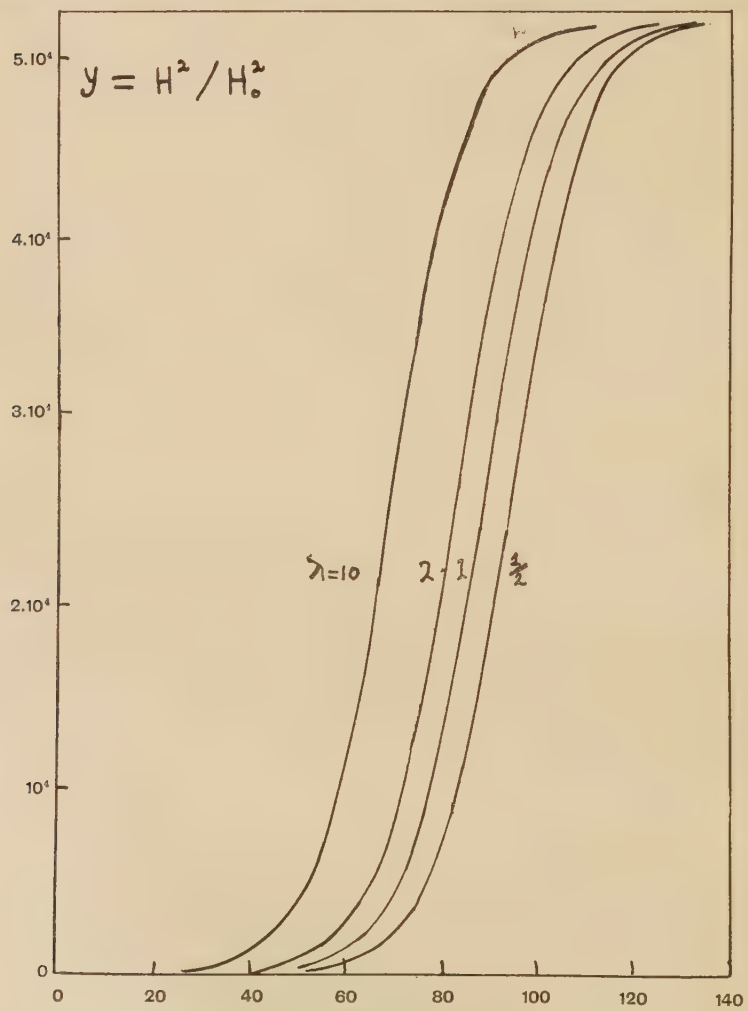


FIG. 5.

unless there are independent electric currents in the layer, which modify the assumed variation. It would be an unexpected coincidence were such currents to be present below but not above the photosphere, for the photospheric level, where opacity sets in, has no special relation to the phenomena here considered. The only independent cause of latitudinal electric currents, apart from the vertical electrostatic field

which helps to produce the drift-currents, is an easterly or westerly electromotive force, which would impel the ions in its own direction, though not the electrons (§ 7), in the region where K is approximately unity. The question whether such an electromotive force actually exists is considered in § 21 ; it is concluded that it is probably absent or of negligible amount, and therefore that H does rise to about 10,000 gauss within a depth of about 100 km. below the photosphere.

The great influence of so thin a layer of the sun's atmosphere upon so intense a magnetic field suggests that there may be important drift-current effects associated with sunspot magnetic fields : I hope to discuss this question in a later paper.

§ 15. So far only the equatorial currents and magnetic field have been considered. Before dealing with their distribution in other latitudes it is convenient to examine the conditions which determine whether a field of given strength proceeding from the sun's interior is practically completely shut in by the drift-currents it produces in the outer layers, or whether it is able to extend outside the sun, though with reduced intensity.

It is sufficient to consider the outer layers as decreasing exponentially in density, corresponding to some " homogeneous height " h ; also, since the main reduction of H occurs in the deeper, denser layers where the drift-current is mainly electronic, it is sufficient to take K = 1 and to regard the current-layer as starting abruptly at the level where $\nu_e = \omega_e$, or $H = 2.10^{-15} n_i$ (*cf.* 16, 17) ; * let this value of H, referring to the base of the current-layer, be denoted by H_b. The field-strength there is due both to the internal field and the drift-currents.

By (28) the total reduction of H^2, that is, $H_b^2 - H_\infty^2$, is approximately equal to $8\pi k n_i T$ (taking K = 1), where n_i refers to the base of the layer. Hence

$$(38) \quad . \quad H_b^2 - H_\infty^2 = (8\pi k T/2 . 10^{-15}) H_b = 9{\cdot}5 . 10^3 H_b.$$

This equation is (approximately) valid only when $H_\infty > 0$; if, according to the equation, $H_\infty < 0$, it implies that the field is reduced to zero before the top of the atmosphere is reached. The maximum value (H_m) of H_b for which $H_\infty = 0$ is evidently $9{\cdot}5 . 10^3$, or really somewhat greater (§ 14), when the exact value of K is taken into account. If H_b much exceeds 10^4 the reduction of the internal field is incomplete.

§ 16. Let H_b' denote the part of H_b due solely to the internal field, the remainder being due to the drift-current layer. The latter part is due to tubes of force that in the absence of this current-layer would extend into outer space ; hence the more complete the shielding by the current-layer, the larger the ratio borne by the second part of H_b to H_b'. For $H_b \leqslant H_m$ the shielding is complete, and the ratio H_b'/H_b is constant ; but this ratio decreases as H_b increases beyond the value H_m.

The distribution of the field just below the current layer is unknown, but it will be assumed that its cause is deeper down, so that in the region just below the current-layer it possesses a potential Ω ; Ω may

* This value depends on the assumed value for the electronic free path : more generally $H = m_e u_e n_i / e a_e$.

be divided into two parts, Ω_i and Ω_0 corresponding to the parts of the field respectively of internal origin, and produced by the current-layer. Then, expanding these two parts of Ω in spherical harmonic functions,

$$(39) \quad \Omega_i = a\Sigma A_n(a/r)^{n+1}P_n,$$

$$(40) \quad \Omega_0 = a\Sigma B_n(r/a)^n P_n,$$

where P_n denotes a Legendre function, and a the radius of the current-layer, which for the present purpose may be treated as a spherical current-sheet of infinitesimal thickness.

If the current-sheet completely shields the outside space from the internal field, no tubes of force cross it, and therefore the radial component of H, which will be denoted by H_r, where

$$(41) \quad H_r = -\frac{\partial\Omega}{\partial r},$$

will be zero at $r = a$. Hence

$$(42) \quad B_n = (n+1)A_n/n,$$

and

$$(43) \quad \Omega = a\Sigma A_n\left\{\frac{n+1}{n}\left(\frac{r}{a}\right)^n + \left(\frac{a}{r}\right)^{n+1}\right\}P_n.$$

The tangential component of H, which will be denoted by H_θ (positive in the direction of increasing north-polar-distance θ), is given by

$$(44) \quad H_\theta = -\frac{\partial\Omega}{r\partial\theta},$$

and at $r = a$, just below the shell, this is

$$(45) \quad -\Sigma\{(2n+1)/n\}A_n dP_n/d\theta.$$

The part corresponding to the harmonic P_n is made up of a fraction $n/(2n+1)$ due to the field of internal origin, and a fraction $(n+1)/(2n+1)$ due to the current-layer: the latter is therefore the greater, and the ratio of the second part to the first is greater, the lower the degree of the harmonic; for P_1 it is 2, *i.e.* only one-third of the whole P_1 term in Ω is of internal origin.

The current-function I for the drift-currents, when the field outside is nil, is

$$(46) \quad I = -\frac{a}{4\pi}\Sigma\frac{2n+1}{n}A_n P_n,$$

and the current density i is

$$(47) \quad i = \frac{\partial I}{a\partial\theta} = -\frac{1}{4\pi}\Sigma\frac{2n+1}{n}A_n\frac{dP_n}{d\theta} = \frac{H_\theta}{4\pi}.$$

§ 17. The condition that the space outside the thin atmospheric shell shall be completely shielded from the internal field depends, as has been seen, on the temperature (and also on other physical conditions,

such as the degree of ionization) of the outer layer. If these are such that the maximum value of H_θ is less than H_m—(*cf.* § 15)—then the atmosphere can nullify this and, *a fortiori*, all lesser values of H_θ. This appears to be substantially the case with regard to the sun. If the maximum value of H_θ had exceeded H_m, the drift-currents would be capable of nullifying H_θ in some latitudes, but not in all; in this case B_n/A_n would not be given by (42), and the part of the internal field which extends outside the shell might differ in character from that within the shell.

Actually the shell is not infinitesimal in thickness, though in comparison with the sun's radius it is very thin. Thus H_θ is only gradually reduced to zero, according to the equations already given, and H_r is not absolutely zero in the layer, because the tubes of force in the higher parts have to pass through the lower ones. The base of the current-layer will probably vary slightly with latitude, being highest in the latitudes where the value of H_θ at the base is least. If the latitude-distribution of H_θ at a given level in the reversing layer were accurately known from observation, it would be possible to calculate the depth of the base of the drift-current layer in each latitude, and the value of H_b as a function of latitude. Conversely, the distribution of H_θ in the reversing layer could be worked out on any assumed hypothesis as to the nature of the internal field, though this problem is more difficult. I hope to give later the result of such a calculation, assuming the presence of only the single harmonic P_1 in Ω_i; it is probable that P_1 is the main term, and that the corresponding value of H_θ at any depth in the reversing layer is proportional to $\sin\theta$, but this is not yet certain.*

The Zeeman effect by which the field is measured is determined by the component H_s of H along the line of sight. When the observer is in the plane of the magnetic equator this component will be $H\cos\theta$, if the small radial component of the field be ignored; according to the above discussion, it is not unlikely that $H = H_\theta = H_e \sin\theta$, where H_e is the equatorial value of H at the level considered. Hence the effective component H_s is $H_e \sin\theta\cos\theta$. This varies with latitude in precisely the same way as in the case of the external field of a uniformly magnetised sphere, viewed by an observer in the equatorial plane, except that here $H_s' = 3H_e' \sin\theta\cos\theta = \frac{3}{2}H_p' \sin\theta\cos\theta$, where H_p' is the polar intensity ($= 2H_e'$) at the given level. Apart from the influence of the very small radial component, which has been ignored in calculating H_s, it would be impossible to distinguish between the two types of field were it not for the obliquity of the sun's equator to the ecliptic, and the slight obliquity of its magnetic to its rotational axis.

The Mount Wilson observations are consistent with the mean relation $H_s \propto \sin\theta\cos\theta$, and the separate analyses which have been made of the observations up to 10° latitude, and beyond, have shown that H is not distributed as over a uniformly magnetised sphere; it is desirable that these observations be reduced again, taking, as the

* The observed fact that the line-of-sight component is approximately proportional to $\sin\theta\cos\theta$ indicates that H_θ is nearly proportional to $\sin\theta$.

theoretical basis, the distribution $H_\theta = H_e \sin\theta$, $H_r = 0$ (very nearly). The variation of tilt of the sun's magnetic axis as viewed from the earth is probably sufficient to show whether H increases two-fold from equator to pole, as on the uniform-magnetisation theory, at the same time declining from the horizontal to the downward vertical, or whether it decreases from H_e to (almost) zero, remaining horizontal. The calculations of *Mount Wilson Communication* No. 61 suggest that H decreases, as is here suggested, from the equator, because the equatorial observations correspond to a value of H_p nearly twice as great as those between 10° and 45° latitude; but this result cannot be strictly interpreted until the adopted basis of calculation, which was shown by the results not to be valid, is replaced by one that gives results consistent with itself.

The Mount Wilson observations of H_s have been expressed in terms of H_p, the supposed polar intensity; on the present theory the maximum intensity of the field occurs at the equator. Comparing the two interpretations of H_s, it appears that $H_e = \frac{3}{2}H_p'$, *i.e.* the maximum intensity corresponding to the measured Zeeman effect is 50 per cent. greater than was deduced on the interpretation hitherto adopted, the value at the equator being 75 instead of 50 gauss.

The radial decrease of the field is probably best determined for the latitude of 45°, where H_θ is $75/\sqrt{2}$ or 53 gauss; the previous estimated lower limit of p_0, the electron or ion pressure at the lowest observed level, is appropriate to this value of H, and therefore to this latitude. But unless p_0 is about twice as great as this lower limit (*i.e.* unless p_0 is about $2 \cdot 10^{-4}$ atmos.) H cannot be 75 at the same level at the equator, and yet decrease to practically zero at the top of the reversing layer. If it does decrease to zero at the equator, then either p_0 is as stated, or, if it is less, it implies that at the same density-level H does not vary as $\sin\theta$; this may possibly be the case (even if Ω beneath the layer involves only the one harmonic P_1) owing to the variation of depth of the drift-currents in different latitudes, already alluded to. Should the rediscussion of the data of *Mount Wilson Communication* No. 61 indicate a divergence from the suggested law of variation of H with latitude, or should the value $2 \cdot 10^{-4}$ of p_0 involve difficulties in the theory of the equilibrium of the solar atmosphere, it will be necessary to consider in detail the vertical distribution of the drift-currents in different latitudes.

§ 18. It is convenient at this stage to verify, in a more definite way, the earlier conclusion that H_r/H_θ is small. It will be assumed that H_r/H_θ increases steadily from the equator, where by symmetry it is zero, to the poles, where by symmetry it is infinite (*i.e.* $H_\theta = 0$ at the poles). Then the tubes of force which cross a sphere of radius r, lying in the current-layer, between the equator and colatitude θ, must cross the equatorial plane between the level r and some higher level r', depending on θ. The equation expressing this fact is

$$(48) \quad \cdot \quad \cdot \quad 2\pi \int_r^{r'} H_e r dr = 2\pi \int_\theta^{\pi/2} H_r r^2 \sin\theta d\theta.$$

Now if, as the observations suggest, H_e decreases to practically zero in the reversing layer, its variation with height is approximately according to the exponential law $\exp(-x/2h)$, where x denotes height above the level r. Hence (48) becomes

$$(49) \quad \int_{\theta}^{\pi/2} H_r \sin\theta d\theta = (2h/r)(H_e - H_e') \quad .$$

where H_e refers to the level r, and H_e' to r'. The left-hand side is equal to the mean value $\bar{H}_r$ of H_r from latitude 0 to $\frac{1}{2}\pi - \theta$; hence

$$(50) \quad \bar{H}_r = (2h/r)(H_e - H_e'). \quad .$$

Since r' increases, and H_e' decreases, the broader the belt of latitude over which the mean value of H_r is taken, it follows that the mean value increases with the breadth of the belt—as is otherwise obvious. The maximum value of $\bar{H}_r$ is got by taking the mean over the whole hemisphere, when $H_e' = 0$. Thus the mean value of H_r over the hemisphere at any level in the current-layer is approximately equal to $2h/r$ times the equatorial intensity at this level. Since $2h = 17$ km., and $r = 6{\cdot}9 \cdot 10^5$ km., it is clear that H_r must be very small compared with H_e or $H_e \sin\theta$ $(= H_\theta)$, except quite near the poles, where H_θ tends to zero. An alternative method of calculation suggests that H_r is of order $(h/r)H_e \cos\theta$, in agreement with the above. It is clear that in the thin current-layer it is legitimate to use the expression $\partial H_\theta/\partial r$ in place of curl H, in order to connect H and i.

§ 19. It has recently been remarked by Ross Gunn * that the spiral motion of the electrons or ions of an ionised atmosphere in the presence of a magnetic field renders the medium diamagnetic, that is, equivalent to a distribution of intensity of magnetisation, directed opposite to H at each point; the magnetisation is

$$(51) \quad knT/H, \quad .$$

where n is the number of charges free to execute the spiral motion. In the lower reversing layer only the electrons are free, so that $n = n_e$ $(= n_i$ according to the approximate assumption here made). Since H is approximately horizontal, the layer is everywhere tangentially magnetised, and such a distribution is equivalent to a distribution of magnetic matter of density σ given by

$$(52) \quad -\frac{1}{\sin\theta}\frac{d}{ad\theta}\left(\frac{knT}{H}\sin\theta\right); \quad .$$

if $H = H_e \sin\theta$, this vanishes, and the diamagnetism produces no external field. Even if the conditions were less special, however, the diamagnetism is much less important than the drift-currents; the magnetic force tangential to and close to the layer, due to a harmonic term $\sigma_n P_n$ in σ, is $4\pi\sigma_n(dP_n/d\theta)/(2n+1)$; if $n = 1$, this is of order

* R. Gunn, *Physical Review*, 1928 July.

$\frac{4}{3}\pi\sigma_1$; now σ_1 is of order $kn\mathrm{T}/a\mathrm{H}$, so that the change $d\mathrm{H}$ in H due to a layer of thickness dr is of order

$$(53) \quad . \quad . \quad . \quad . \quad kn\mathrm{T}dr/a\mathrm{H}.$$

Thus the ratio of the diamagnetic to the drift-current change in H is of order $(kn\mathrm{T}/a) \div m_i gn$, or $k\mathrm{T}/mga$; this is less than 10^{-5}.

§ 20. It has been seen in § 14 that the magnetic intensity (in the equatorial plane) probably increases to about 11,300 at a depth of about 120 km.; below this level, where the drift-currents cease to flow, the potential is probably approximately given by

$$(54) \quad . \quad . \quad . \quad -a\mathrm{A}_1\left\{\frac{2r}{a}+\frac{a^2}{r}\right\}\cos\theta,$$

where $\mathrm{A}_1 = 3800$. The radial component of force is zero (or, more accurately, it is small) at this radius a, and increases rapidly downwards, so long as (54) is valid; the approximate expression for it, based on (54), is

$$(55) \quad . \quad . \quad . \quad -2\mathrm{A}_1\frac{a^3-r^3}{r^3}\cos\theta;$$

the north component $-\mathrm{H}_\theta$ or $\partial\Omega/r\partial\theta$ is

$$(56) \quad . \quad . \quad . \quad . \quad \mathrm{A}_1\left(2+\frac{a^3}{r^3}\right)\sin\theta.$$

The term $2r/a$ in (54) is due to the drift-currents; the other is due to some unknown cause, which must almost certainly consist of electric currents round the magnetic axis. These are likely to be maintained by electromotive forces induced by an internal circulation in the sun, in conjunction with the existing magnetic field. That is, the field is able to maintain itself owing to the presence of this internal circulation; this was suggested in 1919 by Sir J. Larmor.* The important part of the circulation, in this respect, is that which occurs in meridian planes, and, in particular, the component of motion normal to the local magnetic force.

§ 21. Bjerknes† has recently discussed the internal circulation of the sun; he considers that it will tend to become stratified, and that in the outermost circulating layer the motion in the upper part of the layer is equatorward, with a return poleward current beneath. This conclusion is based on the observed variation of mean latitude of sunspots in the course of each 11-year cycle. He ventures the guess (p. 26) that the probable depth to which this first circulating layer extends is between 100 and 1000 km. The equatorward current near the surface, in conjunction with the radial component of force, will induce eastward electromotive forces. These will impel an eastward electric current, which will render the downward increase of magnetic force still greater than has been calculated in § 10, by considering the

* Sir J. Larmor, *Brit. Assoc. Report*, 1919, p. 159.
† V. Bjerknes, *Mount Wilson Contributions*, No. 312.

drift-currents alone. It is possible to make a rough estimate of the induced current-intensity in the drift-current layer; the mean meridianal velocity v of the gas corresponding to a passage from latitude 45° to the equator in 11 years is approximately 110 cm./sec.; if σ is the electrical conductivity of the medium, the eastward current-intensity due to the induced electromotive force $v\mathrm{H}_r$ is $\sigma v\mathrm{H}_r$. According to the calculations made in my note on the electrical conductivity of stellar matter,* the value of σ in the drift-current layer, taking the temperature as 6000°, and allowing for the fact that the electrons do not contribute appreciably to σ in this region, is approximately $1{\cdot}7 \cdot 10^{-11}$; hence $\sigma v = 2 \cdot 10^{-9}$ approximately, and, taking $\mathrm{H}_r = (2h/r)\mathrm{H}_\theta$, which by (50) is correct as regards order of magnitude (though H_r depends rather on $\cos\theta$ than, like H_θ, on $\sin\theta$), we have for the order of magnitude of i

$$(57) \quad . \quad . \quad 2 \cdot 10^{-9}(2h/r)\mathrm{H}_\theta \quad \text{or} \quad 5 \cdot 10^{-14}\mathrm{H}_\theta.$$

Considering the decrease of H_θ which these induced currents alone would produce, using the formula $d\mathrm{H} = -4\pi i dr$, we have

$$(58) \quad . \quad . \quad . \quad \frac{d \log \mathrm{H}_\theta}{dr} = -6{\cdot}3 \cdot 10^{-13}.$$

In a layer 100 km. thick, $\log \mathrm{H}_\theta$ decreases by $6{\cdot}3 \cdot 10^{-13} \cdot 10^{7}$ or $6{\cdot}3 \cdot 10^{-6}$; the corresponding decrease of H_θ is quite negligible, showing that such induced currents, even in so thick a layer, are negligible compared with the drift-currents, which can reduce H from 50 to 10 in about 20 km.

§ 22. Below the drift-current layer the electrons contribute to σ, which is thereby increased 270-fold, apart from the further increase due to the gradually increasing temperature. Moreover, H_r increases rapidly as we go inwards, according to (55). All these facts will render the deeper meridianal motions more effective in inducing currents than are the corresponding motions in the outer layer: it may be, therefore, that the return circulation towards the poles, which will produce a westerly electromotive force, can induce a current-sheet of the strength necessary to maintain the internal field Ω_i. This return circulation belongs not only to the outer circulating stratum, but also to the stratum below, in which the circulation is opposite. The product of v into the thickness of the two (poleward and equatorward) streams in each circulating stratum must be the same, so that the total electric current density integrated through the thickness of each stream is proportional to the mean value of $\sigma\mathrm{H}_r$ in each stream; as already stated, this is much greater in the poleward than in the outer equatorward stream of the outer circulating layer. The value of H_r will depend on the depth at which the poleward stream is situated.

A rough calculation suggests that the internal field may be produced by an electric current-sheet of relatively small thickness, say 1000 km. To explain the field of internal origin, for which H_θ at the base of the

* *M.N.*, **89**, 54.

drift-current layer is about 3800 at the equator, it is necessary for the current-strength (integrated through the thickness of the induced current-sheet) to be about $3800/2\pi$ or 600 e.m.u. (*i.e.* 6000 amperes per cm. length along the meridian) if the current-sheet is not too far below the drift-current layer. The corresponding current-density, for a layer 1000 km. or 10^8 cm. thick, is $6 \cdot 10^{-6}$, which must equal $\sigma v H_r$; at this level σ is of order 10^{-8}, so that vH_r must be of order $6 \cdot 10^2$. If $(a - r)$ is of order 1000 km., by (55) H_r will be of order 40, requiring v to be of order 15 cm./sec. In our present ignorance of the internal circulation of the sun, these quantities do not seem unreasonable, though the assumed thickness of the circulating strata is perhaps too large. Owing to the downward increase of σ, a deeper layer would be more effective if the value of v was of the same order of magnitude.

The calculation of the *distribution* of induced currents in the poleward stream, and of the internal field, is somewhat difficult: the same difficulties occur in connection with the circulation theory when applied, as Sir Joseph Larmor suggested, to the explanation of the earth's magnetic field; I hope later to publish some calculations I have made on this problem.

§ 23. If this theory of the sun's internal field be correct, it seems necessary to suppose that the internal circulation is not perfectly symmetrical about the axis of rotation of the sun's outer layers, but forms a system symmetrical about the magnetic axis, and revolving, with this axis, in 31·8 days about the axis of rotation. I am not aware of any other evidence afforded by the surface phenomena of the sun, apart from its magnetic field, in favour of this rather startling conclusion.

Summary.

1. It is shown that the combined effect of the gravitational, electrostatic, and magnetic fields existing in the sun's reversing layer will be to produce an eastward "drift-current" of electrons, and that this current is of the right order of magnitude to explain the radial limitation of the sun's magnetic field. The theory sets a lower limit to the pressure at the base of the reversing layer, which is of the same order as, but slightly greater than, the value assigned on other grounds by E. A. Milne. The reduction of the magnetic intensity from 50 to 10 gauss probably occurs in a layer only about 25 km. thick.

2. It is shown that the observed rapid downward increase of intensity of the sun's magnetic field probably continues to a depth of about 100 km. below the photosphere, where the intensity rises to about 10,000 gauss.

3. The drift-currents in the sun's upper layer merely confine within the atmosphere tubes of magnetic force of internal origin, which would otherwise spread out into space. The internal field, the intensity of which, just below the drift-current layer, is between 3000 and 4000 gauss, must be produced by a system of westward electric currents in a deeper layer: the nature of the field between this region and the drift-current layer is described. The westward current system is attributed

to electromagnetic induction by internal circulation of solar matter in meridian planes ; it is shown that the orders of magnitude involved in this theory are not unreasonable, if the circulation is similar to that postulated by Bjerknes in his theory of the sunspot cycle.

Addendum.

The determination of H_b, the magnetic intensity at the base of the drift-current layer in the sun's atmosphere, depends on the value assumed for nl, where n is the number of electrons or ions per c.c., and l is the mean free path of an electron. The value adopted above (§ 6), $1{\cdot}4 \cdot 10^{15}$, seems on further consideration likely to be too high. The mean free path has a precise meaning only when the particles concerned are rigid elastic spheres ; in other cases some artificial diameter d has to be used in the formula $1/l = \sqrt{2}\pi n d^2$. One method of determining d (which here refers to the sum of the effective radii of an ion and electron) is by comparing the formulæ for the coefficient of diffusion D_{12}, derived on the hypothesis that the particles behave like rigid elastic spheres, and on the correct hypothesis of attraction according to the inverse square law. Applying this method to the solar atmosphere, we find $d = 7 \cdot 10^{-7}$ cm., leading to $nl = 4{\cdot}5 \cdot 10^{11}$. If this value were correct, the drift-currents would not enable the solar atmosphere to confine the sun's magnetic field within itself, as it is observed to do. But there is reason to think that the collisions between ions and electrons may be more effective in preventing ordinary diffusion than in destroying the drift-velocity of the electrons, since the latter is constantly re-created by the magnetic, electrostatic, and gravitational fields after each collision.

Another method of estimating d is to take it to be the perpendicular distance between the ion and the relative path of an electron before their collision, corresponding to some chosen limit of the angle δ of deflection of the relative motion of the electron. For an ionized gas at 6000°, the values of d which give $\delta = 60°$, $\delta = 90°$ are $1{\cdot}6 \cdot 10^{-7}$ cm. and $4{\cdot}7 \cdot 10^{-8}$ cm. ; the corresponding values of nl are $8{\cdot}8 \cdot 10^{12}$ and $1{\cdot}0 \cdot 10^{14}$. The latter is perhaps too large, and the former too small ; their ratios to the value assumed in my paper are 1/160 and 1/14. Whatever the right ratio may be, H_b must be reduced in the same proportion, and therefore probably to less than 1/14 the value (about 11,000 gauss) given above. The reduction will make it easier to account for the maintenance of the sun's field by internal circulation in meridian planes. The range of uncertainty in nl shows the need for the treatment of the problem by the complicated, exact methods of the kinetic theory ; this investigation is in progress.

The Sun's General Magnetic Field and the Chromosphere.
By S. Chapman, M.A., D.Sc., F.R.S.

§ 1. In the preceding paper (which will be referred to by the letter R) the proposed theory of the limitation of the sun's magnetic field is worked out on the assumption that the solar atmosphere is uniform in temperature T, and supported by ordinary gas-pressure. In the upper reversing layer and chromosphere selective radiation-pressure becomes important; it is here shown how this affects the theory, and, further, it is shown that the strong eastward motion in the chromosphere affords evidence that the general magnetic field of the sun cannot extend appreciably into the chromosphere.

§ 2. For definiteness, the chromosphere will be taken to begin at the level where the electron-pressure p_e is 10^{-12} atmos. or 10^{-6} dynes/cm.^{2}. Taking $T = 4800°$, the number n_e of electrons per c.c. at this level is $1{\cdot}5 \cdot 10^6$. At height x above this level it is supposed that

$$(1) \qquad n_e = 1{\cdot}5 \cdot 10^6 x_0^2/(x + x_0)^2,$$

where $x_0 = 2000$ km. $= 2 \cdot 10^8$ cm. approximately. Below this level there is the transition layer where, as we descend, selective radiation-pressure decreases in importance relative to the gas-pressure. For simplicity it is supposed that the only other kind of particle present is singly ionised calcium, either in the normal or excited state; the normal ions are supposed to receive an upward impulse $m_i u$, at the end of each interval τ, taken to be $5 \cdot 10^{-5}$ sec.

§ 3. The rate of upward decrease of n_e is far slower than that of a neutral gas of molecular mass m_e, and the electrons are held down, as elsewhere in the atmosphere, by an (upward) electrostatic field E. The equation of partial pressure for the electrons is

$$(2) \qquad \frac{\partial p_e}{\partial x} = -n_e(m_e g + e\mathrm{E}) = n_e \mathrm{X}_e;$$

for the ions,

$$(3) \qquad \frac{\partial p_i}{\partial x} = -n_i(m_i g - e\mathrm{E} - \mathrm{X}_r) = n_i \mathrm{X}_i;$$

where X_r denotes the upward force of selective radiation-pressure, here treated as continuous, though in reality it is a succession of individual impulses; in § 8 the legitimacy of this approximation is established.

Taking $n_e = n_i$, it follows that $p_e = p_i$, and $\mathrm{X}_e = \mathrm{X}_i$; the small term $m_e g$ in X_e can be neglected, so that we may write

$$(4) \qquad -e\mathrm{E} = \mathrm{X}_e = \mathrm{X}_i = -m_i g + e\mathrm{E} + \mathrm{X}_r;$$

also, from (2), if T may be treated as constant,

$$(5) \qquad e\mathrm{E} = -k\mathrm{T}\frac{\partial \log n_e}{\partial x} = \frac{2k\mathrm{T}}{x + x_0};$$

hence

$$(6) \quad . \quad . \quad X_r = m_i g - 2eE = m_i g - \frac{4kT}{x + x_0}.$$

The upward impulse $m_i u$ must equal $X_r \tau$, or

$$(7) \quad . \quad . \quad . \quad u = g\tau - \frac{4kT}{x + x_0}\frac{\tau}{m_i}.$$

§ 4. The drift velocity V due to the vertical force X is

$$(8) \quad . \quad . \quad . \quad \frac{X}{eH}\frac{\omega^2}{\nu^2 + \omega^2} = \frac{X}{eH}\frac{1}{1 + R^2/l^2},$$

(*cf.* R, § 5), and the eastward current-intensity i is given by

$$(9) \quad . \quad . \quad . \; i = n_i KX/H = (K/H)\partial p_i/\partial x.$$

Assuming R (2), we have

$$(10) \quad . \quad . \quad . \quad . \quad \frac{dH^2}{dx} = 8\pi K\frac{\partial p_i}{\partial x},$$

or, so long as K is approximately constant,

$$(11) \quad . \quad . \quad . \quad H^2 = \text{constant} + 8\pi K p_i,$$

the result already proved in R, but on more restricted assumptions.

§ 5. The factor K cannot exceed 2, so that the total decrease of H^2 in passing upwards right through the chromosphere cannot exceed $16\pi p_0$, or $5 \cdot 10^{-5}$: the square root of this is $7 \cdot 10^{-3}$ gauss or 700γ, so that if H_c, the value of H at the base of the chromosphere, exceeds this value, the field must extend beyond the chromosphere into the corona.

It is not possible to say forthwith that if $H_c < 700\gamma$ the magnetic field must be reduced to zero within the chromosphere; the actual value of R has here to be considered. Since $R_i/l_i = 3 \cdot 10^{-12} n_i/H$, $R_e/l_e = 1{\cdot}6 \cdot 10^{-15} n_i/H$, and since in the chromosphere $n_i < 1{\cdot}5 \cdot 10^6$, therefore $3 \cdot 10^{-12} n_i < 4{\cdot}5 \cdot 10^{-6}$, so that if $H < 5\gamma$, or $5 \cdot 10^{-5}$ gauss, K is practically 2. Any field less than 5γ will here be called negligible. Hence for any appreciable field ($>5\gamma$) K = 2, and it follows that for $H_c < 700\gamma$ the field must be reduced to zero within the chromosphere. In the last stage of its reduction, where it becomes less than 5γ, K may be reduced below 2, but the final disappearance of this remnant of the field takes place in a very thin layer (*cf.* R (36)).

§ 6. Wherever the magnetic field in the chromosphere is appreciable, the ions are free to take up the drift-velocity $V_i = X/eH$, or E/H, equal to $2kT/e(x + x_0)$, or $8{\cdot}3 \cdot 10^7/H(x + x_0)$. Since $x + x_0 > x_0 = 2{\cdot}10^8$ cm., $V_i < 0{\cdot}41/H$ cm./sec.; for example, if $H = 700\gamma$, $V_i = 60$ cm./sec., while if $H = 5\gamma$, $V_i = 84$ metres/sec. These velocities are relative to the magnetic field, which has a synodic rotation period of 31·8 days, corresponding to an equatorial speed of 1·61 km./sec. St. John and Babcock * find that the linear equatorial speed for the

* C. E. St. John and H. D. Babcock, *Mt. Wilson Contributions*, No. 278, 1924; *Ap. J.*, **60**, 32.

lower reversing layer is 1·97 km./sec., and that the speed increases upwards to 2·18 km./sec. for high levels in the chromosphere. The latter represents a velocity of ·57 km./sec. relative to the magnetic field. This cannot be interpreted as a drift-velocity in the above sense; for suppose, in the first place, that the ions are free (or R_i/l_i small), so that

$$\cdot 57 \cdot 10^5 = \frac{8\cdot 3 \cdot 10^7}{H(x+x_0)},$$

then $H = 1\cdot 5 \cdot 10^3/(x+x_0) < 1\cdot 5 \cdot 10^3/2 \cdot 10^8 < 10^{-5}$, or $H < 1\gamma$; for such a value of H the ions are not free to take up the full drift-motion, *i.e.* R_i/l_i is not small. The complete formula for V_i must therefore be used, *i.e.*

$$(12)\quad V_i = \frac{1}{1+(R_i/l_i)^2}\frac{2kT}{eH(x+x_0)} = \frac{2kT}{e(x+x_0)}\frac{1}{H+9\cdot 10^{-24}n_i^2/H}.$$

For any assigned value of x (and therefore of n_i) the last factor is greatest when $H = 3 \cdot 10^{-12}n_i$, or $R_i = l_i$; to this corresponds

$$(13)\quad V_i = \frac{2kT}{e(x+x_0)}\frac{1}{6\cdot 10^{-12}n_i} = 2\cdot 3 \cdot 10^{-4}(x+x_0).$$

For this to be $\cdot 57 \cdot 10^5$, $x+x_0$ must be $2\cdot 5 \cdot 10^8$, corresponding to $n_i = \cdot 96 \cdot 10^6$, and $H = 2\cdot 9 \cdot 10^{-6}$ or $\cdot 29\gamma$, while $K = 1\frac{1}{2}$; the level is 500 km. above the base of the chromosphere. But this minute field would be rapidly reduced in a very thin layer: consider the thickness of the layer in which it is reduced in the ratio 1/200, at which point $R_e/l_e = 1$ and $K = \frac{1}{2}$. The thickness of this layer is over-estimated by taking $K = \frac{1}{2}$ throughout; since

$$(14)\quad dH^2 = 8\pi K dp_i = -\frac{16\pi K p_0 x_0^2 dx}{(x+x_0)^3},$$

we have

$$(15)\quad dx < \frac{(2\cdot 9 \cdot 10^{-6})^2(x+x_0)^3}{16\pi\frac{1}{2}p_0x_0^2};$$

inserting $x+x_0 = 2\cdot 5 \cdot 10^8$, $x_0 = 2\cdot 10^8$, $p_0 = 10^{-6}$, we find $dx < 1\cdot 3 \cdot 10^2$ cm. Thus in a layer less than 1·3 metre thick H is reduced from $\cdot 29\gamma$ to $\cdot 0015\gamma$, and in a few more metres thickness it is 10^{-10} or less.

§ 7. Thus the observed eastward motion of the chromosphere cannot be explained as due to a drift in the above sense, for magnetic fields of the necessary weakness could exist only throughout a layer of negligible thickness. If there were an appreciable magnetic field in the chromosphere, the eastward motion would necessarily be much less rapid than is observed. It seems probable that the sun's general magnetic field dies out in the reversing layer at a level such that R_i/l_i is never small, *i.e.*, the ions are never free to have their motion governed by the magnetic field, and K never much exceeds 1.

The polar plumes of the corona have often been supposed to indicate, or at least to suggest, the presence of a magnetic field in that region, though the form of the corona in lower latitudes does not support the

suggestion, but seems rather to be controlled by prominences or other disturbance-phenomena at lower levels. It would appear from the above that if there be a magnetic field above the chromosphere, it cannot be an upward extension of the general magnetic field observed at low levels, but must have an independent origin.

The absence of a magnetic field in the chromosphere, apart from the local fields associated with sunspots, is of importance in connection with the theory of the emission of clouds or streams of ionised particles, such as are supposed to produce terrestrial magnetic storms. Such clouds or streams might be prevented from departing far from the sun if the general magnetic field extended beyond the chromosphere.

§ 8. The above proof that there is no general magnetic field in the chromosphere depends on the assumption that, in calculating the drift-velocity, the succession of radiative impulses can be treated as a continuous force ; this point will now be examined. It will be supposed that the impulses are all vertically upward, and the random impulses of the light-quanta emitted by the excited ions on reversion to the normal state will be ignored : the following proof can, however, easily be generalised to take them into account.

Between successive random upward impulses $m_i u$, the motion of the ions is governed by the (vector) equation

$$(16) \quad . \quad . \quad . \quad . \quad m_i \dot{v} - ev\,\mathrm{H} = \mathrm{X},$$

where $v\,\mathrm{H}$ denotes the vector product, and $\mathrm{X} = m_i g + e\mathrm{E}$. It is sufficient to ignore the motion along the field, so that v may be regarded as a vector in the xy plane, and it can conveniently be represented as a complex quantity (x being the real and y the imaginary axis). The solution of the equation is

$$(17) \quad . \quad . \quad . \quad v = (i\mathrm{X}/e\mathrm{H}) + v' \exp(-i\omega t),$$

where X, eH, $\omega(= e\mathrm{H}/m_i)$, t are real, and v, v' complex ; v' is arbitrary so far as the differential equation is concerned.

At the end of each interval $\tau(= 5 \cdot 10^{-5}$ second), v is suddenly increased by the amount u, after which its variation proceeds according to (17), with the modified value of v'. Let t be reckoned from the instant of a material collision, and let $v_0, v_1, v_2, \ldots v_0', v_1', v_2', \ldots$ be the values of v and v' at times $t = 0$, $t = \tau$, $t = 2\tau$, . . . just *after* the impulses received at these instants. Then, writing $\omega\tau = \theta$,

$$v_n' = v_{n-1} \exp(-i\theta) + u, \qquad v_n = i\mathrm{X}/e\mathrm{H} + v_n' ;$$

by successive substitution it may be seen that

$$\begin{aligned} v_n &= \frac{i\mathrm{X}}{e\mathrm{H}} + v_0' e^{-ni\theta} + u(1 + e^{-i\theta} + e^{-2i\theta} + \ldots + e^{-(n-1)i\theta}) \\ &= \frac{i\mathrm{X}}{e\mathrm{H}} + v_0' e^{-ni\theta} + u\frac{1 - e^{-ni\theta}}{1 - e^{-i\theta}}. \end{aligned}$$

The motion according to these equations is terminated by a material collision, at which the ion receives an impulse far larger than those received from light-quanta. In the chromosphere material collisions

occur at intervals of duration, on the average, $1/\nu_i = 1/7{\cdot}4 \,.\, 10^{-10} n_i$, or about 1000 seconds. Thus n, the total number of radiative impulses received in this time, is a very large number, and liable to fluctuate considerably, even for slight variations in the material-collision-interval. Hence, in calculating the average value of v for an ion over the period between material collisions we may regard $\exp(-ni\theta)$ as a random vector with zero average value. We thus get

$$\begin{aligned} \bar{v} &= (i\mathrm{X}/e\mathrm{H}) + u/(1 - \exp - i\theta) \\ &= (i\mathrm{X}/e\mathrm{H}) - iu/\theta, \end{aligned}$$

since θ is very small. Substituting for X and u, we have

$$\begin{aligned} \bar{v} &= \frac{i}{e\mathrm{H}}\left(\mathrm{X} - \frac{m_i u}{\tau}\right) \\ &= \frac{i}{e\mathrm{H}}(- m_i g + e\mathrm{E} - \mathrm{X}_r) = \frac{i\mathrm{X}_e}{e\mathrm{H}}. \end{aligned}$$

This is the value of the drift-velocity already found by treating the radiation-pressure as continuous.

Magnetic Storms and Solar Activity 1874 *to* 1927.

By W. M. H. Greaves, M.A., and H. W. Newton.

(*Communicated by the Astronomer Royal.*)

1. The object of this paper is to present the results of a comparison of the Greenwich solar and magnetic observations over the period 1874 April (the date at which systematic solar observations were commenced) to 1927 December. As has been explained in a previous paper,* a catalogue of storms was prepared, including all magnetic disturbances with ranges equal to or greater than 30′ for Declination or 150γ for Horizontal Force (or North Force for the years 1916 to 1925) or Vertical Force. The total number of storms thus catalogued is 403.† It is proposed to publish this list in the volume of *Greenwich Results* for 1927.

In the earlier paper referred to a special list of 60 great storms (ranges equal to or greater than 1° in Declination, or 300γ in either of the other elements) was compared with a catalogue of " naked-eye " sunspots, and it was found that a storm was associated with the presence of a large spot near the central meridian of the sun more often than could be ascribed to chance ; moreover, that this association of storms with spots became more definite if comparison was restricted to a smaller list of storms consisting of the very largest storms of all.

The present paper gives the results of a discussion of the remaining 343 smaller storms. The greater storms previously considered will

* *M.N.R.A.S.*, **88**, 556, 1928.

† Given in the previous paper as 405. In two cases two storms in the first list have subsequently been reckoned as one.

again appear in the discussion, and it will be seen that the conclusions of the previous paper are confirmed by a comparison of spot activity at the times of great and small storms.

2. *Distribution of Magnetic Disturbances in the Solar Cycle.*—Table I. shows the number of greater and smaller storms which occurred in each year during the period 1875 to 1927 inclusive. In forming the totals given in columns 4 and 8, each "great" storm was counted as two. The totals are plotted in the figure, and for comparison the usual curve giving the Greenwich mean daily areas corrected for foreshortening is given at the top. The two curves exhibit a general similarity, and in particular there is a close correspondence of the minima.

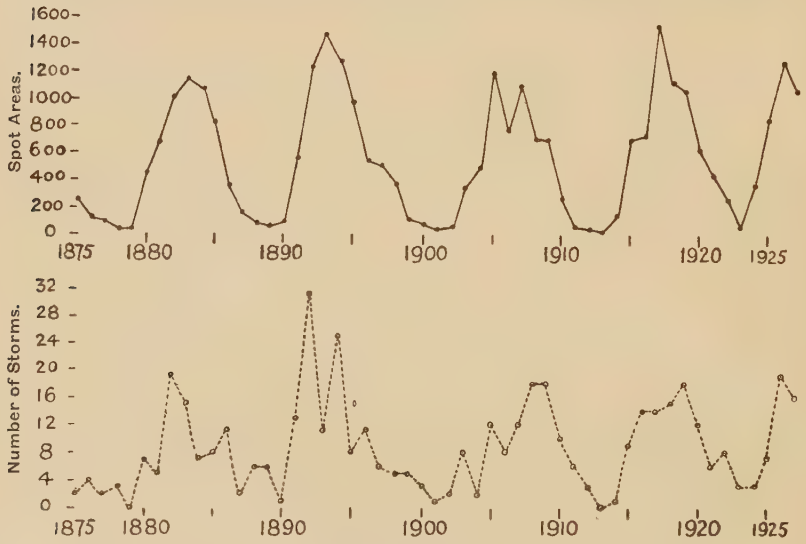
TABLE I.

Year.	Great Storms.	Smaller Storms.	Total.	Year.	Great Storms.	Smaller Storms.	Total.
1875	0	2	2	1902	0	2	2
76	0	4	4	03	2	4	8
77	0	2	2	04	0	2	2
78	0	3	3	05	0	12	12
79	0	0	0	06	0	8	8
1880	1	5	7	07	1	10	12
81	2	1	5	08	2	14	18
82	5	9	19	09	3	12	18
83	2	11	15	1910	0	10	10
84	1	5	7	11	0	6	6
85	2	4	8	12	0	3	3
86	1	9	11	13	0	0	0
87	0	2	2	14	0	1	1
88	0	6	6	15	1	7	9
89	0	6	6	16	0	14	14
1890	0	1	1	17	2	10	14
91	1	11	13	18	1	13	15
92	9	13	31	19	1	16	18
93	1	9	11	1920	2	8	12
94	8	9	25	21	1	4	6
95	0	8	8	22	0	8	8
96	0	11	11	23	0	3	3
97	0	6	6	24	0	3	3
98	2	1	5	25	0	7	7
99	0	5	5	26	4	11	19
1900	1	1	3	27	3	10	16
01	0	1	1				

A diagram such as this shows quite clearly that magnetic storms tend to occur at times of sunspot maximum and to avoid periods of sunspot minimum, but it does not necessarily follow that magnetic disturbance is associated with increased spot activity at the time

of disturbance. In order to decide whether this is so or not, a further investigation is necessary.

3. *Analysis of Spot Activity at the Time of Magnetic Storms.*—For this purpose, in the case of each storm, the spots, which were within four days (53° of heliographic longitude) either side of the central meridian of the sun at the time of the commencement of the storm, were examined, and the spot of largest mean area being selected, its mean area * was written down against the storm. Furthermore, the mean of the ranges in D, H (or N), and V was written down as a measure of the strength of the storm (the D range being converted into force for this purpose). The 343 smaller storms were then divided into groups



Frequency of Sunspots and Magnetic Storms at Greenwich 1875 to 1927.
(Limits of storms : D $\geqslant 30'$; H, N, or V $\geqslant 150\gamma$.)

according to their intensity, and the mean values of the mean spot areas, selected as explained above, were formed for each group. Of the 60 " great " storms discussed in the previous paper, 43 form another group ; and the remaining 17 " great " storms, consisting of those known to possess ranges greater than or equal to 90′ in D or 500γ in either other element, form a further group called " Very Great Storms." †

For purposes of comparison the corresponding mean areas for a group of magnetically " quiet " days are given. This group of days was obtained by selecting the two days in each month of smallest total character figure from the De Bilt circulars over the period 1914 to 1924 inclusive, *i.e.* over one complete solar cycle.

* The mean area, as tabulated in the *Greenwich Photoheliographic Results*, is the mean of the areas (corrected for foreshortening) on each successive day as the spot crosses the disc.

† See Table II. of previous paper.

The mean areas for the groups are given in the second and fourth columns of Table II. The mean areas in these columns are not directly comparable, as those in the central zone refer to a zone of 106° heliographic longitude, and those near the limb refer to a zone of 52° heliographic longitude. The third and fifth columns, however, give the relative values of the numbers in the second and fourth columns, the result for " quiet " days being taken as unity in each case.

TABLE II.

Mean Areas of Largest Sunspots at Time of Commencement of Magnetic Storms.

Limits of Storm.	Spots near Centre of Disc 53° E. to 53° W.		Spots near Sun's Limbs 53° to 79° E. and W.		Number of Storms.
	Mean Area.	Relative Values.	Mean Area.	Relative Values.	
" Quiet days "	244	1·0	167	1·0	264 (quiet days)
$<120\gamma$	260	1·1	181	1·1	42
$120–150\gamma$	273	1·1	159	1·0	122
$150–180\gamma$	329	1·3	220	1·3	91
$180–210\gamma$	390	1·6	302	1·8	52
$>210\gamma$	462	1·9	189	1·1	36
Great storms	691	2·8	251	1·5	43
Very great storms	1116	4·6	181	1·1	17

Areas are corrected for foreshortening.

It seems quite clear that as we proceed to the greater storms there is a tendency for larger spots to be found near the central meridian of the sun at the time of the commencement of the storm, and that there is no definite tendency of this nature near the limbs.

The figures do indicate a slight increase in the spots near the limbs at the time of storms as compared with the " quiet " days, but this may be due to the tendency (shown in the figure) for storms to occur at times of sunspot maximum.

We have thus found a relationship between the occurrence of magnetic storms and the state of the central region of the sun at the times of the storms. It will be observed, however, that the relationship consists of an increase in the spot areas for the larger storms, and that there is no significant difference between smaller storms and " quiet " days. This point is of importance in view of attempts which have been made to investigate the relation of magnetic disturbance and solar activity from statistical studies of solar phenomena on " disturbed " and " quiet " days, *i.e.* on the most disturbed and least disturbed days of each month. In the period discussed here there would be 3225 disturbed days (5 to each month), but there are only 403 storms down to the limit of our Catalogue. Clearly, in comparing " disturbed " and " quiet " days, we are in effect contrasting the latter to disturbances which on the average are of the same order of activity as the smallest storms grouped in Table II. A glance at the table

shows that nothing very significant could be expected to come out of such an analysis, more especially when mean spot areas or sunspot numbers relating to the state of the whole disc are used as a basis for investigation.

As an example of the kind of results one might expect to obtain by comparison of solar conditions on " quiet " and " disturbed " days, we may quote a result obtained by Chree,* who, dealing with the eleven years 1890–1900, tabulated the Greenwich daily projected areas of spots on the ten quietest and ten most disturbed days of each month. The final means gave an excess of about 8 per cent. in spot area for the " disturbed " days over the " quiet " days. In the present investigation, by concentrating attention on the larger storms, we have found an excess in the spot areas at the times of the largest storms as compared with the quiet-day periods of the order of 300 or 400 per cent.

The results may be taken as confirming and extending the conclusion reached in the previous paper that large storms are associated with large spots more often than can be ascribed to chance.

4. With a view to investigating still further the portion of the sun's disc within which spot activity might be related to the occurrence of magnetic storms, Table III. has been prepared. For this purpose six zones were taken on the visible hemisphere, each zone consisting of 26° of heliographic longitude. For each storm the mean area of the spot of largest mean area existing in each zone at the time of the commencement of each storm (or on one of the selected " quiet " days) was written down against the storm, and the storms were then grouped and the areas meaned as before. It is only in the third, fourth, and fifth columns that any significant increase of spot area can be detected as we pass from the moderate to the greatest storms. This suggests that the occurrence of large magnetic storms on the earth is related to the presence of large spots in a zone of the sun extending from about longitude 26° E. to 53° W. of the sun's central meridian at the time of the commencement of the storm.

TABLE III.

Mean Areas of Largest Sunspots in Zones of Heliographic Longitude (Zero = Longitude of Central Meridian at Commencement of Storm).

Limits of Storm.	E. 79°–53°.	E. 53°–27°.	E. 26°–0°.	W. 0°–26°.	W. 27°–53°.	W. 53°–79°.	Number of Storms.
" Quiet days "	85	79	82	89	91	109	264 (quiet days)
$<120\gamma$	93	125	61	79	100	111	42
$120–150\gamma$	78	77	141	95	75	105	122
$150–180\gamma$	117	98	119	93	137	127	91
$180–210\gamma$	92	64	170	144	121	247	52
$>210\gamma$	70	74	153	276	152	133	36
Great storms	115	119	203	306	308	176	43
Very great storms	122	172	460	380	445	78	17

* *Phil. Trans.*, (A), **208**, 238, 1908.

5. *Magnetic Storms on Spotless Days.*—But whilst we are thus led to the conclusion that the larger magnetic storms originate in some form of solar activity which is at any rate statistically connected with the outbreak of sunspots, it does not seem feasible to regard the spots as necessarily the actual seat of origin of the storms. There are, in point of fact, a number of storms commencing on spotless days. These storms are given in Table IV. The letters "S.C." denote a typical "sudden commencement." None of these storms had a mean range of as much as 210*γ*; in six cases the mean range exceeded 180*γ*.

TABLE IV.

Magnetic Storms on Spotless Days.

Storm.	Commencement G.C.T.	Storm.	Commencement G.C.T.
1877 May 28–29	May 28·8 S.C.	1902 Aug. 21–22	Aug. 21·6
1878 May 14–15	May 14·3 S.C.	1908 Mar. 26–Apr. 1	Mar. 26·5
1886 Nov. 2–6	Nov. 2·7	1909 Aug. 28–Sept. 1	Aug. 28·7 S.C.
1886 Nov. 29–Dec. 3	Nov. 29·6	1910 June 20–22	June 20·1
1887 Sept. 25–28	Sept. 25·7	1910 Aug. 21–22	Aug. 21·5
1887 Nov. 21–22	Nov. 21·4	1910 Aug. 28–30	Aug. 28·6
1888 Apr. 11–14	Apr. 11·2	1911 Jan. 24–27	Jan. 24·5
1888 June 3–6	June 3*	1911 Feb. 21–24	Feb. 21·4
1888 Oct. 30–Nov. 1	Oct. 30·8 S.C.	1911 Mar. 20–29	Mar. 20·0 S.C.
1889 Oct. 5–8	Oct. 5·7	1911 May 14–16	May 14·7
1889 Nov. 1–3	Nov. 1·2	1911 Oct. 10–11	Oct. 10·7
1889 Nov. 26–29	Nov. 26·6	1912 Aug. 5–6	Aug. 5·7
1897 Apr. 23–25	Apr. 23·3 S.C.	1912 Oct. 14–16	Oct. 14·7
1900 Mar. 12–14	Mar. 12·9	1921 Oct. 7–9	Oct. 7·8
1901 May 10	May 10·3 S.C.	1922 Jan. 24–26	Jan. 24·5
1902 Apr. 10–11	Apr. 10·4 S.C.	1924 Jan. 29–30	Jan. 29·6

These 32 storms, several of which appear during prolonged spotless periods, negative any theory which insists that the source of origin of a storm must necessarily be a sunspot. One must accordingly turn elsewhere for the origin, and it is natural to examine solar phenomena taking place at higher levels in the sun's atmosphere. The Greenwich data include measures of faculæ, and an examination of these is at once suggested. On direct photographs faculæ are visible only near the limb, but their heliographic longitudes having been measured when thus seen, it is possible to deduce the position of facula areas near the centre of the disc at any time. In this way it was found that for 30 out of the 32 storms in Table IV. there were some faculæ within four days of the central meridian at the time of the commencement of the storm. As regards the size of these faculæ, the majority

* The time of commencement of this storm cannot be given owing to loss of trace.

were of small or moderate extent such as are found on the sun at practically any time. The two exceptions when no faculæ were found in the central region of the disc are the storms of 1889 November 1–3 and 1901 May 10. With respect to the former, there were small faculæ of total area 166 millionths of the hemisphere near the west limb at the time of commencement of the storm, but in the latter case not only were there no spots but there were no faculæ anywhere on the disc. In fact, there were no spots or faculæ measured on the Greenwich series from 1901 March 28 to May 18 inclusive.

6. *Magnetic Storms with "Sudden Commencements."*—In forming the catalogue of storms, the practice was adopted of noting whether a storm showed a typical "sudden commencement." The movements known as "sudden commencements" * are characteristic of a large class of storms which seem to commence almost simultaneously all over the earth, thereby suggesting that this class of storm is caused by the impingement on the earth of a cloud or stream of particles thrown off from the sun. So far as the storms which have been classed as "Great Storms" are concerned, most of them (54 out of 60) possess the "S.C." characteristic, and inasmuch as it is these great storms which have been found to be connected with spot activity, it seems worth while to inquire whether the relationship exhibited by Table I. would be improved by confining the analysis to the "S.C." storms.

To test this suggestion, Tables V. and VI. have been prepared. They are in all respects similar to Table II. It will be seen on comparing Table V. with Table II. that the run of the areas of the central spots with the intensities of the storms is rather more marked for the "S.C." storms. With respect to Table VI., there is indeed no obvious correlation, but it must be remembered that there are only six "great" storms here, and it is the "great" storms which give us the certainty of reality in the relationship which appears in Table II.

Table V.

Mean Areas of Sunspots and Magnetic Storms with "Sudden Commencements."

Limits of Storm.	Spots near Centre of Disc 53° E. to 53° W.		Spots near Sun's Limbs 53° to 79° E. and W.		Number of Storms.
	Mean Area.	Relative Values.	Mean Area.	Relative Values.	
"Quiet days"	244	1·0	167	1·0	264 (quiet days)
$<120\gamma$	339	1·4	228	1·4	16
$120–150\gamma$	333	1·4	210	1·3	38
$150–180\gamma$	425	1·7	283	1·7	29
$180–210\gamma$	596	2·4	307	1·8	21
$>210\gamma$	525	2·2	211	1·3	20
Great storms	721	3·0	218	1·3	37
Very great storms	1116	4·6	181	1·1	17

* The chief characteristic of a "sudden commencement" is a rapid movement in H which marks the beginning of the storm. See C. Chree, *Studies in Terrestrial Magnetism*, chap. xii.

TABLE VI.

Mean Areas of Sunspots and Magnetic Storms without "Sudden Commencements."

Limits of Storm.	Spots near Centre of Disc 53° E. to 53° W.		Spots near Sun's Limbs 53° to 79° E. and W.		Number of Storms.
	Mean Area.	Relative Values.	Mean Area.	Relative Values.	
"Quiet days"	244	1·0	167	1·0	264 (quiet days)
$<120\gamma$	211	0·9	153	0·9	26
$120–150\gamma$	246	1·0	136	0·8	84
$150–180\gamma$	284	1·2	190	1·1	62
$180–210\gamma$	252	1·0	298	1·8	31
$>210\gamma$	386	1·6	163	1·0	16
Great storms	508	2·1	455	2·7	6
Very great storms	..	..	..	..	..

The available data certainly suggest that it is the storms with "S.C.s" which are responsible for the sunspot relationship, but it is impossible with the existing amount of material to be dogmatic on this point.

7. *Summary and General Discussion.*—It has been found that on the average at the times of large magnetic storms larger sunspots will be found near the centre of the sun's disc than is the case at the times of smaller disturbances. There is no similar tendency for larger spots to be found near the limbs at the times of large storms, and no significant difference in spot activity has been found between the smallest storms and magnetically quiet days.

This is a statistical result, and it is not the case that large spots will invariably accompany the outbreak of a large magnetic storm. In fact, attention has been drawn in a previous paper * to the case of a very large storm (that of 1894 November 13–14) which was accompanied by only very small spots. The results suggest that magnetic disturbances have their origin in some solar disturbance which will be situated in the central portion of the disc at the time of the outbreak of the storm, and that if the solar disturbance is capable of giving rise to a large magnetic storm there will be a tendency for it to be accompanied by a large spot. That the spots themselves, however, are not to be regarded as the origin of magnetic disturbances is shown by the occurrence of storms during spotless periods. In particular, attention has been directed to the storm of 1901 May 10. There were no spots or faculæ measured at Greenwich between 1901 March 28 and May 18.

These conclusions are in accord with the theory, advanced by Maunder and developed by Chapman, which ascribes magnetic storms to the impingement on the earth of some agency proceeding from a restricted area of the sun's surface and travelling outwards in a limited

* *Loc. cit.*, p. 560.

stream. The data set forth above in section 4 suggest that the source of disturbance will, at the time of commencement of a large storm, be situated in a zone lying between two days east and four days west of the sun's central meridian, the most probable position being about one day west. The large storms usually last for about a day, and so, if we assume the most probable direction of the centre of the stream to be normal to the sun's surface, we get an interval of the order of one and a half days as the time taken by the agency to travel from the sun to the earth.*

As regards the nature of the agency, Lindemann † has suggested that it consists of clouds of ionised gas ejected by radiation-pressure from the sun. E. A. Milne ‡ has investigated the circumstances under which ionised atoms can be expelled from the chromosphere, and his work suggests that an ionised positively charged stream produced in this way would take about four days to travel from the sun to the earth. The results obtained in this paper suggest, however, that the time-interval from sun to earth is less than this, unless, indeed, there is a systematic tendency for streams to be projected from the seat of disturbance in the direction of the sun's rotation.

It is thus suggested that the solar disturbances which are responsible for magnetic storms are situated in the chromosphere. The present discussion indicates that these solar disturbances possess the following characteristics:—

(*a*) Their frequency follows the eleven-year cycle;

(*b*) There is a tendency for the larger of such disturbances to be accompanied by sunspots.

It may be noticed that metallic prominences possess the properties (*a*) and (*b*).§ An extended examination of the relationship between such prominences and magnetic storms is accordingly suggested. A study of the metallic prominences and other chromospheric phenomena occurring at the times of the larger storms (such as those listed in Table I. of our previous paper) would be useful.

There seem to be grounds for suspecting that storms with "sudden commencements" are more closely correlated with sunspots than storms without this characteristic. The available data are, however, inadequate to decide this point definitely.

* The same conclusion was reached in the previous paper, p. 564. See also Maunder, *M.N.R.A.S.*, **64**, 206, 1904, who obtained a mean interval of a day between the commencement of a storm and the central meridian passage of the associated spot. Abetti gets a similar result (*Atti della Reale Accademia Nazionale dei Lincei*, **5**, 725, 1927).

† *Phil. Mag.*, **38**, 684, 1919.

‡ *M.N.R.A.S.*, **86**, 459, 1926.

§ J. and M. A. Evershed, *Memoirs of the Kodaikanal Observatory*, **1**, pt. ii., 72–76, 1917.

Royal Observatory, Greenwich:
1928 *October* 20.

On the Frequency Distribution of Restricted Proper Motions.

By W. M. Smart, M.A., D.Sc.

(*Communicated by the Director, Cambridge Observatory.*)

Summary.

In any region of the sky the distribution, according to position angle, of proper motions limited to a particular minimum value is distinct from the distribution when no restriction is imposed. With Dyson's Law of stellar density, the effect of this restriction on the character of the true drift-curves can be derived; the results of the calculations are given in Table II. In practical applications, the restricted distribution can hardly be differentiated from a true drift-curve with a velocity, however, considerably greater than the real velocity of the complete drift of stars. The systematic motions of the stars (with proper motions exceeding 20″ per century) discussed by Dyson twenty years ago are examined in relation to the theoretical implications of the paper.

§ 1. In the majority of researches on the systematic motions of the stars, no restriction is made as to the magnitudes of the proper motions. In such investigations the observed distribution of proper motions in position angle can be confronted with theoretical drift-curves, with each of which three constants are associated, namely, the number of stars forming the drift, the drift-velocity, and the direction in the sky in which the drift is pointing. The theoretical curves are independent of the law of distribution of the stars in space, for the only thing that matters is the direction in which a particular star is moving, and it is immaterial whether the resultant proper motion is small or great. We can state this more precisely as follows. If $h\mathrm{V}$ is the drift-velocity and w the linear velocity of a star (the resultant of its *motus peculiaris* and the drift-velocity)—both projected on the tangent plane of the celestial sphere at the centre of the region considered—and if there are N stars forming the drift which is pointing in position angle θ_1, then the number of stars with linear velocities between w and $w + dw$ and in position angles between θ and $\theta + d\theta$ is

$$\frac{\mathrm{N}h^2}{\pi} we^{-h^2(w^2+\mathrm{V}^2-2\mathrm{V}w\cos\overline{\theta-\theta_1})}d\theta dw.$$

Further, if $\rho d\theta$ is the number of stars with proper motions *between zero and infinity* in position angles between θ and $\theta + d\theta$, then ρ is given by

$$\rho = \frac{\mathrm{N}h^2}{\pi}\int_0^\infty we^{-h^2(w^2+\mathrm{V}^2-2\mathrm{V}w\cos\overline{\theta-\theta_1})}dw$$

in which the linear velocities w can have all values between zero and infinity, and no special knowledge of the distribution of the stars in

space need be presumed. In Eddington's notation,* we can write ρ in the form

$$\rho = \frac{N}{2\sqrt{\pi}} e^{-h^2V^2} f(\tau) \quad . \quad . \quad . \quad (1)$$

where

$$\tau = hV \cos(\theta - \theta_1) \quad . \quad . \quad . \quad . \quad (2)$$

But when we restrict the values of the proper motions μ in some definite way, we require to know the law of stellar distribution before we can derive a theoretical frequency curve with which the proper motions so restricted can be compared. In a later communication, the proper motions measured by Dr. V. Nechvile,† from plates taken at Paris, will be examined in this connection; with insignificant exceptions, Nechvile's proper motions have a lower limit of 2″ per century, proper motions below this limit not having been measured.

§ 2. In a recent paper ‡ I have considered Dyson's proposed law of stellar distribution in connection with proper motion statistics. Assuming that the volume density $D(r)$ of the stars at a distance r in a given region follows the law

$$D(r) = \frac{A}{r} e^{-h^2k^2r^2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

I have derived the frequency law of proper motions restricted to values of μ and over in the form

$$N(\mu, \theta) = \frac{N}{2\sqrt{\pi}} e^{-h^2V^2} \{f(\tau) - \eta^2 f(\eta\tau)\} \quad . \quad . \quad (4)$$

where N is the total number of stars in the region with *all* values of μ, $N(\mu, \theta)d\theta$ is the number of proper motions of μ and over between position angles θ and $\theta + d\theta$, and η depends on the limiting value μ and on the value of k, being given by

$$\eta = \frac{\mu}{(\mu^2 + k^2)^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad (5)$$

The series of frequency curves given by (4), when plotted for any combination of the constants involved, have a marked resemblance to the series of frequency curves, of much simpler analytical form, given by (1), and in practical applications the difference between one of the curves (4) and a particular member of the series of curves (1), defined by a new value hV_1—which we may describe as the pseudo drift-velocity of the restricted frequency distribution of proper motions—would almost certainly be difficult of detection. But the number of stars in the region with proper motions of μ and over is $\int_0^{2\pi} (N\mu, \theta)d\theta$

* *M.N.*, **68**, 588.
† *Publications de l'Observatoire National de Prague*, No. 4.
‡ *M.N.*, **88**, 567.

or $R(\mu)$, and it has been shown * that $R(\mu)$ is related to N (the total number of proper motions from 0 to ∞) by

$$R(\mu) = N\{1 - \eta^2 e^{-h^2V^2(1-\eta^2)}\} \quad . \quad . \quad . \quad (6)$$

Let us assume then that the observed frequency distribution of the restricted proper motions is identified with a theoretical curve of the form given by (1), the pseudo drift-velocity being denoted by hV_1. Then

$$N(\mu, \theta) = \frac{R(\mu)}{2\sqrt{\pi}} e^{-h^2V_1^2} f(\tau_1) \quad . \quad . \quad . \quad (7)$$

where

$$\tau_1 = hV_1 \cos(\theta - \theta_1) \quad . \quad . \quad . \quad (8)$$

For obvious reasons, the value of θ_1 is the same in (8) as in (2).

The value of hV_1 will clearly depend on the minimum value μ of the proper motions considered and, of course, on the value of k in the density function. The practical problem consists firstly in deriving hV_1 from the observed distribution of the restricted proper motions as represented by the theoretical distribution given by (7), and, secondly, in deducing the value hV of the true drift-velocity.

§ 3. From equations (4), (6), and (7) we have

$$F(\eta, \tau) \equiv \frac{e^{-h^2V^2}\{f(\tau) - \eta^2 f(\eta\tau)\}}{1 - \eta^2 e^{-h^2V^2(1-\eta^2)}} = e^{-h^2V_1^2} f(\tau_1) \quad . \quad (9)$$

If hV_1 is obtained from the observations, this equation enables us to calculate the true drift-velocity hV, it being assumed that the value of k—and consequently from (5) the value of η—is known. But it is much simpler to invert the process and to calculate the value of the pseudo drift-velocity hV_1 corresponding to definite values of the true drift-velocity and of k; from these results it is then easy to deduce, in any particular instance, the value of hV from the observed value of hV_1. In the calculations which follow, the minimum value of μ is taken to be 2″ per century, so that the results may be applicable at once to Nechvile's proper motions; three values of hV are considered (0·5, 1·0, and 1·5) in combination with each of the values 2·0, 1·0, 0·5, and 0·0 of the constant k in the density function.

It is readily seen that if η tends to 1, the numerator and denominator of the left-hand side of (9) each tend to zero; therefore for values of η in the neighbourhood of 1 a special adaptation of (9) is necessary. But when η is not greater than about 0·9 the equation (9) can be thrown into a form suitable for direct calculation. Writing

$$e^{-h^2V^2} f(\tau) = \phi(hV, \theta - \theta_1) \quad . \quad . \quad . \quad (10)$$

equation (9) becomes

$$\frac{\phi(hV, \theta - \theta_1) - \eta^2 \phi(hV, \theta' - \theta_1)}{1 - \eta^2 e^{-h^2V^2(1-\eta^2)}} = \phi(hV_1, \theta - \theta_1) \quad . \quad (11)$$

* *M.N.*, **88**, 580 [Equation (17)].

where θ' is defined by

$$\cos(\theta' - \theta_1) = \eta \cos(\theta - \theta_1).$$

The values of the function ϕ on the left-hand side of (11) can be written down—at 10° intervals, say, in the value of $\theta - \theta_1$—by inspection from the table which I have given in an earlier paper.* As, however, the practice in dealing with the frequency curves of proper motions is to smooth the curves by taking the sums in three adjacent 10° sectors, the values of ϕ, in the table referred to, have been treated in the same way. (The values of ϕ in the table refer to counts in 10° sectors.)

In Table I. are the results of the calculations when the true drift-velocity is taken to be 0·5 and η to be 0·7 (corresponding to $k = 2{\cdot}0$, the minimum proper motion being 2″ per century); in the columns headed A are the computed values of the left-hand side of (11) at 10° intervals in the values of $\theta - \theta_1$.

TABLE I.

$\theta-\theta_1$	A.	B.	$\theta-\theta_1$.	A.	B.
°			°		
0	4·03	4·04	90	1·20	1·20
10	3·96	3·96	100	0·99	1·00
20	3·75	3·73	110	0·81	0·85
30	3·42	3·37	120	0·69	0·73
40	3·01	2·94	130	0·60	0·64
50	2·57	2·51	140	0·53	0·57
60	2·13	2·12	150	0·48	0·52
70	1·76	1·76	160	0·42	0·49
80	1·44	1·45	170	0·41	0·48

In the columns headed B are the values of ϕ corresponding to $hV_1 = 0{\cdot}6$ (these have been derived from the tabulated values of ϕ by taking the sums of three adjacent 10° sectors). The close accordance between the values in the columns A and B indicate sufficiently clearly that in this instance at least the omission of all proper motions less than 2″ per century from a drift of stars with a drift-velocity 0·5 leads to a frequency curve with the almost exact characteristics of a true drift-curve with the pseudo drift-velocity 0·6. In other words, the distribution of the restricted proper motions suggests a drift-velocity of 0·6 which is in excess by 0·1 of the true drift-velocity derived from *all* the stars in the region whatever the values of their proper motions may be.

§ 4. When η has values in the neighbourhood of 1, we adapt equation (9) as follows: Write

$$\eta = 1 - \alpha \quad . \quad . \quad . \quad . \quad (12)$$

* *M.N.*, 87, 128.

in which α is presumed to be small. Up to terms in α^3, we have

$$f(\tau) - \eta^2 f(\eta\tau) = \alpha\left\{2f(\tau) + \tau\frac{df}{d\tau}\right\} - \alpha^2\left\{f(\tau) + 2\tau\frac{df}{d\tau} + \tfrac{1}{2}\tau^2\frac{d^2f}{d\tau^2}\right\} + \alpha^3\left\{\tau\frac{df}{d\tau} + \tau^2\frac{d^2f}{d\tau^2} + \tfrac{1}{6}\tau^3\frac{d^3f}{d\tau^3}\right\} \quad (13)$$

Also the function $f(\tau)$ is defined * by the equation

$$\frac{\sqrt{\pi}}{2}f(\tau) = \tau e^{\tau^2}\mathrm{I}(\tau) + \tfrac{1}{2}$$

where

$$\mathrm{I}(\tau) = \int_{-\tau}^{\infty} e^{-x^2}dx$$

so that

$$\frac{d\mathrm{I}(\tau)}{d\tau} = e^{-\tau^2}.$$

From these we have

$$\tau\frac{df}{d\tau} = (1 + 2\tau^2)f(\tau) - \frac{1}{\sqrt{\pi}}$$

$$\frac{d^2f}{d\tau^2} = (6 + 4\tau^2)f(\tau) - \frac{2}{\sqrt{\pi}}$$

$$\tau\frac{d^3f}{d\tau^3} = (6 + 24\tau^2 + 8\tau^4)f(\tau) - \frac{1}{\sqrt{\pi}}(6 + 4\tau^2).$$

Inserting these in (13), we have

$$e^{-h^2V^2}\{f(\tau) - \eta^2 f(\eta\tau)\} = A\alpha - B\alpha^2 + C\alpha^3$$

where

$$A = e^{-h^2V^2}\left\{(3 + 2\tau^2)f(\tau) - \frac{1}{\sqrt{\pi}}\right\}$$

$$B = e^{-h^2V^2}\left\{(3 + 7\tau^2 + 2\tau^4)f(\tau) - \frac{1}{\sqrt{\pi}}(2 + \tau^2)\right\}$$

$$C = e^{-h^2V^2}\left\{(1 + 9\tau^2 + 8\tau^4 + \tfrac{4}{3}\tau^6)f(\tau) - \frac{1}{\sqrt{\pi}}(1 + 3\tau^2 + \tfrac{2}{3}\tau^4)\right\}$$

or, in a form more suitable for computation,

$$A = (3 + 2\tau^2)\phi(hV, \theta - \theta_1) - \frac{1}{\sqrt{\pi}}e^{-h^2V^2},$$

$$B = (3 + 7\tau^2 + 2\tau^4)\phi(hV, \theta - \theta_1) - \frac{1}{\sqrt{\pi}}(2 + \tau^2)e^{-h^2V^2},$$

$$C = (1 + 9\tau^2 + 8\tau^4 + \tfrac{4}{3}\tau^6)\phi(hV, \theta - \theta_1) - \frac{1}{\sqrt{\pi}}(1 + 3\tau^2 + \tfrac{2}{3}\tau^4)e^{-h^2V^2}.$$

* A. S. Eddington, *M.N.*, **67**, 36.

In a similar manner, we obtain

$$1 - \eta^2 e^{-h^2V^2(1-\eta^2)} = U\alpha - V\alpha^2 + W\alpha^3 \quad . \quad . \quad (14)$$

where

$$U = 2(1 + h^2V^2),$$

$$V = 1 + 5h^2V^2 + 2h^4V^4,$$

$$W = 4h^2V^2 + 6h^4V^4 + \tfrac{4}{3}h^6V^6.$$

We thus find that (11) can be written

$$F(\eta, \tau) \equiv \frac{A - B\alpha + C\alpha^2}{U - V\alpha + W\alpha^2} = \phi(hV_1, \theta - \theta_1) \quad . \quad (15)$$

The values of the coefficients A, B, . . . W can be easily calculated, and the values of the function $F(\eta, \tau)$ found at intervals, say, of 10° in $\theta - \theta_1$. These values, as we have stated previously, can be associated with the corresponding values for a true drift-curve with a particular drift-velocity hV_1; in this way hV_1 is determined.

In the limit when η is unity (or $\alpha = 0$), the equation (15) reduces to the simple form

$$\frac{(3 + 2\tau^2)\phi(hV, \theta - \theta_1) - \frac{1}{\sqrt{\pi}}e^{-h^2V^2}}{2(1 + h^2V^2)} = \phi(hV_1, \theta - \theta_1) \; . \quad (16)$$

This is the form corresponding to $k = 0$, so that the density function $D(r)$ is given simply by

$$D(r) = \frac{A}{r}.$$

§ 5. In Table II. are given, in heavy type, the values of the pseudo drift-velocity hV_1, corresponding to different values of the true drift-velocity hV and to different values of k; equations (11), (15), and (16) have been used in the calculations according to circumstances. When $hV = 0{\cdot}0$, that is when the centre of the region is identical with the apex or ant-apex of the true drift, the distribution of proper motions is symmetrical and is obviously unaffected by any possible restriction of the proper motions or by the form of the density law.

TABLE II.

Values of hV_1.

	hV.			
k.	0·0.	0·5.	1·0.	1·5.
2·0	**0·0**	**0·60**	**1·10**	**1·58**
1·0	**0·0**	**0·66**	**1·22**	**1·65**
0·5	**0·0**	**0·68**	**1·25**	**1·71**
0·0	**0·0**	**0·70**	**1·26**	**1·73**

Assuming that the observations furnish the value of hV_1, and that the value of k is known, the data of Table II. enable us to deduce the true drift-velocity hV.

Fig. 1 illustrates the results for $hV = 0{\cdot}5$. The curve A is the

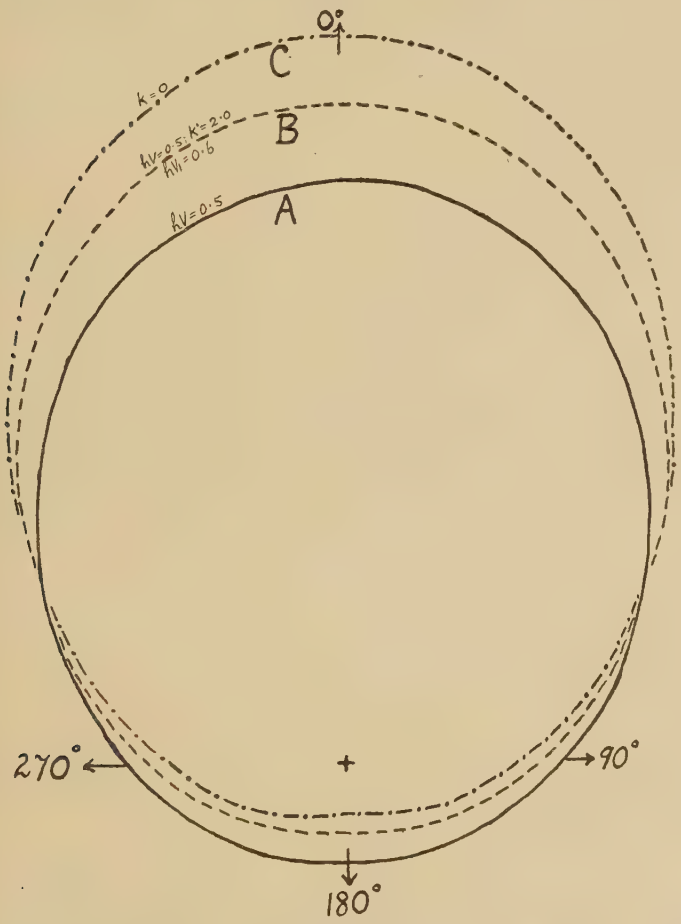


FIG. 1.—Curve A, true drift-curve for $hV=0{\cdot}5$; Curve B, distribution of the restricted proper motions for $k=2{\cdot}0$; Curve C, distribution of the restricted proper motions for $k=0{\cdot}0$.

true drift-curve in polar co-ordinates. Curve B has been computed from equation (11)—the values of the radius vector are to be found in column B of Table I.—the value of k being assumed to be 2·0; it is indistinguishable from a true drift-curve with velocity 0·6. Curve C has been derived from equation (16)—it corresponds to the limiting value of k, namely, 0·0; the differences between this curve and a true

drift-curve with velocity 0·7 are so small that in practical applications C would be identified with the latter. It is to be remembered that curves B and C are based on centennial proper motions of 2″ and over. When k is large in comparison with the minimum value μ of the proper motions, the value of η is small, and it is readily seen from (9) that in this instance the distribution of the restricted proper motions approaches closely to the true drift-curve (A, in fig. 1). Whatever the value of k, the theoretical distribution of the restricted proper motions gives rise to a curve occupying an intermediary position between the true drift-curve and the limiting curve for which $k = 0$ (in fig. 1, between the curves A and C).

§ 6. If we write

$$\Delta = hV_1 - hV,$$

we see from Table II. that for any definite value of k, Δ appears to have a maximum within the range of values of hV with which the table deals; for example, taking the last line (corresponding to $k = 0$), we notice that Δ increases from 0·0 (for $hV = 0·0$) to 0·26 (for $hV = 1·0$) and decreases to 0·23, for $hV = 1·5$. It is of interest, as well as a check on our calculations, to derive a fairly reliable estimate of the value of hV at which Δ is a maximum. This can be done fairly easily for the limiting value of k, i.e. for $k = 0$, as follows. If we identify the maximum of the function given on the left-hand side of (16) with the maximum of a true drift-curve with velocity hV_1, we can write

$$\frac{(3 + 2x^2)\phi(x) - \frac{1}{\sqrt{\pi}}e^{-x^2}}{2(1 + x^2)} = \phi(x_1) \quad . \quad . \quad (17)$$

where

$$x = hV \quad \text{and} \quad x_1 = hV_1.$$

Δ is then defined by

$$\Delta = x_1 - x,$$

and as x_1 is a function of x by (17), Δ is a maximum when $\frac{dx_1}{dx} = 1$. Now

$$\phi(x) = \frac{2}{\sqrt{\pi}}x\mathrm{I}(x) + \frac{1}{\sqrt{\pi}}e^{-x^2},$$

and (17) can then be written

$$2x\mathrm{I}(x) + e^{-x^2} + \frac{x\mathrm{I}(x)}{1 + x^2} = 2x_1\mathrm{I}(x_1) + e^{-x_1^2}.$$

Differentiating this last equation and putting $\frac{dx_1}{dx} = 1$, we obtain, after some reduction,

$$y = y_1$$

where

$$y = \frac{3 + x^2}{x(1 + x^2)^2}\left[\phi(x) - \frac{1}{\sqrt{\pi}}e^{-x^2}\right] + \frac{2x}{1 + x^2}\phi(x) \quad . \quad (18)$$

and

$$y_1 = \frac{2}{x_1}\left[\phi(x_1) - \frac{1}{\sqrt{\pi}}e^{-x_1^2}\right] \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Table III. gives the values of y and y_1 for selected values of x or of x_1.

Table III.

x.	0·6.	0·8.	1·0.	1·2.	1·4.	1·6.
y	12·13	11·77	11·38	10·94	10·76	10·60
y_1	9·38	10·20	10·76	11·00	11·17	11·20

Using these values graphically, it is seen that the curves whose equations are given by (18) and (19) intersect when $x = 1{\cdot}19$—this corresponds to the maximum value of Δ. Thus when $k = 0$, the difference between the pseudo drift-velocity $h\mathrm{V}_1$ and the true drift-velocity $h\mathrm{V}$ is a maximum for $h\mathrm{V} = 1{\cdot}19$, thus verifying the general trend of the entries in Table II. For other values of k, the maximum of Δ would appear to be in the neighbourhood of that just found.

§ 7. The results in my paper * "On the Law of Stellar Distribution Derived from Proper Motions" appear to indicate that the values of k in the density function depend on galactic latitude, and range from 0·0 at the galactic equator to about 2·0 at galactic latitude 60°. It follows that if the proper motions are restricted to say 20″ per century and over (as in a well-known paper † by Sir Frank Dyson), the distribution of proper motions will approximate very closely to the limiting distribution corresponding to $k = 0{\cdot}0$, for then η is very nearly equal to 1. In other words, the large proper motions ought to indicate a pseudo drift-velocity considerably in excess of the true velocity of the drift. Let Δ denote this excess. If λ is the angular distance between the centre of a particular region and the apex (or ant-apex) of the drift, the true space velocity $h\mathrm{W}$ of the drift is given by

$$h\mathrm{W} = h\mathrm{V} \operatorname{cosec} \lambda.$$

If the pseudo drift-velocity is used in the analysis of the systematic motions of the stars, the error in the deduced value of $h\mathrm{W}$ amounts to $\Delta \operatorname{cosec} \lambda$. If λ is 30°, we see from Table II. that the error in $h\mathrm{W}$ may amount to as much as 0·4, which is a considerable fraction of the true values of $h\mathrm{W}$ (1·6 for stream I. and 0·8 for stream II.). Accordingly, in the complete analysis of proper motions restricted to a particular minimum value, the effects of this restriction must first be removed from the apparent drift-velocities. With Dyson's Law of stellar density, these effects can be inferred from the results of this paper.

§ 8. As a test of the theory outlined in the previous pages, I have analysed the proper motions with which Dyson's well-known paper ‡ deals. These proper motions have a limiting minimum magnitude of 20″ per century; accordingly, if the assumed law of stellar density is

* *M.N.*, **88**, 567.

† *Proc. Roy. Soc. Edin.*, **28**, pt. iii., 231.

‡ "The Systematic Motions of the Stars," *Proc. Roy. Soc. Edin.*, **28**, pt. iii., No. 13.

obeyed, the frequency curves of these large proper motions should correspond very nearly to the limiting form of the theoretical curves given by $k = 0$. The analysis gives the values of the pseudo drift-velocity $h\mathrm{V}_1$ for each of the drifts, and the data of Table II. enables us to derive the values of the true drift-velocity $h\mathrm{V}$. Dyson's analysis was made by a graphical process for each of the four regions considered, and he was mainly concerned with the directions in which each of the drifts was pointing. The counts of stars are given for $7\frac{1}{2}°$ sectors; from these I have deduced the counts for 30° sectors for which the table of the numerical values of the theoretical function has been adapted. The results of the analysis are as follows:—

TABLE IV.

	Drift I.			Drift II.		
	$h\mathrm{V}_1$.	θ_1.	R(μ).	$h\mathrm{V}_1$.	θ_1.	R(μ).
		°			°	
Region A (North and South Poles)	1·6	175	202	0·4	5	120
Region B (R.A. 0^h : Dec. 0°)	1·9	$97\frac{1}{2}$	164	1·1	190	157
Region C (R.A. 4^h : Dec. 0°)	1·1	120	125	1·3	185	125
Region D (R.A. 8^h : Dec. 0°)	1·0	260	63	1·4	165	156

As before, R(μ) denotes the number of stars forming the pseudo drift. The values of θ_1 in Table IV. show little alteration from Dyson's values (in his paper the position angle θ_1 has not a uniform definition for all the regions). The feature of the table is the high values of $h\mathrm{V}_1$ for each of the two drifts in the regions where either drift would be expected to show the greatest prominence (that is, when the centre of the region is about 90° from the apex of a drift). Dyson's deduced values for the positions of the apices are in close accordance with other determinations. Using these, we derive the values of the angular distance (λ) of the centres of the regions from each of the two apices. The pseudo drift-velocities $h\mathrm{V}_1$ have been corrected by means of the data in Table II.; the deduced values of $h\mathrm{V}$ (correct to 0·05) are given in Table V. below. The space velocity of a drift is given by

$$h\mathrm{W} = h\mathrm{V} \operatorname{cosec} \lambda,$$

and the values of $h\mathrm{W}$ derived from each region are given in the fourth and seventh columns of Table V. The values of $h\mathrm{W}$ obtained provide a test of the theory.

TABLE V.

	Drift I.			Drift II.		
Region.	$h\mathrm{V}$.	λ.	$h\mathrm{W}$.	$h\mathrm{V}$.	λ.	$h\mathrm{W}$.
		°			°	
A	1·4	97	1·4	0·25	164	0·9
B	1·7	94	1·7	0·75	98	0·75
C	0·85	33	1·55	1·05	106	1·1
D	0·75	27	1·65	1·15	98	1·15

The mean values of $h\mathrm{W}$ for Drift I. is just under 1·6 and for Drift II. just under 1·0. The values found by Professor Eddington from the

analysis of the Boss stars (without any restriction as to the magnitude of the proper motions) are 1·52 and 0·86 respectively. The agreement for Drift I. is as nearly perfect as could be expected; the result for Drift II. is, however, somewhat higher than Eddington's value. In Dyson's regions C and D, Drift II. is unusually prominent, as even a casual glance at the diagrams on p. 237 of his paper will show; and no reasonable change in the adopted position of Drift II. apex (normally, this apex is not so well determined as that of Drift I.) will serve to modify the values found for regions C and D. On the whole, the agreement, as regards the deduced values of the space velocities of the drifts, between the theory of this paper and the distribution of the large proper motions, is quite satisfactory.

§ 9. At first sight, it may seem that an additional test can be applied by means of the observed numerical distribution of the stars between the two drifts. Many investigations into the systematic motions of the stars show that the numerical strengths of the two star streams are not far removed from equality. Table IV. gives the numbers $R(\mu)$ of stars with centennial proper motions of 20″ and over; equation (6) enables us to calculate the numbers N in the true drifts. Now from (5),

$$\eta = \frac{20}{(400 + k^2)^{1/2}},$$

or, with sufficient accuracy,

$$\eta \equiv 1 - \alpha = 1 - \frac{k^2}{800}.$$

Hence from (6) and (14)

$$R(\mu) = CNk^2(1 + k^2V^2) \quad . \quad . \quad . \quad (20)$$

in which C is a constant. Assuming that k is the same for Drift I. as for Drift II., we can now calculate the ratio of N_1 to N_2, the numbers of stars forming the true drifts. Using the values of kV in Table V. and expressing N_1 and N_2 as percentages of $N_1 + N_2$, we obtain the results in Table VI.

Table VI.

Region.	N_1.	N_2.
A	38	62
B	30	70
C	55	45
D	38	62

These figures appear to show that Drift II. is numerically stronger than Drift I. in the ratio 3 : 2 approximately, which is at first sight a rather unexpected result.

There can be no doubt that some of the stars with which Dyson's paper deals belong to the class of high velocity stars studied by Oort and Strömberg; as such they should have been excluded from our analysis. The phenomenon was, of course, unknown twenty years

ago, and with the available data it is hardly possible to make a precise estimate of the effect produced by the inclusion of the high velocity stars in the observed frequency curves. Oort's apex * of motion, relative to the sun, for these stars is $7^{h}\cdot8$, $-43°$, and their motions are directed towards the hemisphere of which this point is the pole. Using the position angles of the drift apices and the position angles of Oort's apex for the four regions, we find that for region A the high velocity stars would be likely to affect the observed number of stars in Drift I., and hardly at all the number of stars in Drift II.; in region D the opposite is the case; in regions B and C the expectation is that the effect on each drift would be much about the same. On a balance, the ratio of 3 : 2 is probably a fair approximation to the ratio of the true drift stars, on the supposition that all the high velocity stars have been excluded.

We can look at this result from another point of view. The mean distance r of a group of stars distributed according to Dyson's density law is given by

$$\bar{r} = \frac{\int_0^\infty r^2 e^{-h^2k^2r^2}dr}{\int_0^\infty re^{-h^2k^2r^2}dr}$$

or

$$\bar{r} = \frac{\sqrt{\pi}}{2hk} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

in terms of the theoretical units adopted. It is unlikely that the stars of Drift I. are at exactly the same mean distance as the stars of Drift II.; in other words, from (21), k_1 and k_2 (the values of k for Drift I. and Drift II. respectively) can hardly be expected to be equal. Equation (20) shows that the ratio $N_1 : N_2$ can be made unity (to agree with the general results of observation) by assuming that $k_2^2 = \frac{3}{2}k_1^2$, or that $k_1 = 1\cdot2k_2$. The mean distances $\bar{r}_1$ and $\bar{r}_2$ of the stars in Drift I. and Drift II. are then by (21) connected by the relation

$$\bar{r}_1 = 1\cdot2\bar{r}_2.$$

Although no great dependence need be placed on the numerical accuracy of this result, qualitatively it would appear to be in accordance with Professor Eddington's results † for the Boss stars—the mean parallaxes for stars of Drift I. and Drift II. were found to be $0''\cdot0156$ and $0''\cdot0166$ respectively.

It is hoped to make a more detailed investigation, on the lines indicated in §§ 8, 9, from the extensive proper motion material which has accumulated since the date of Sir Frank Dyson's paper.

* *Groningen Publications*, No. 40, p. 69.
† *Stellar Movements*, p. 116.

Observatory, Cambridge:
1928 *October* 2.

On Schwarzschild's Ellipsoidal Theory. By W. M. Smart, M.A., D.Sc.

(*Communicated by the Director, Cambridge Observatory.*)

§ 1. *Summary.* (*a*) In any region in which there are N stars with proper motions ranging from zero upwards, the theoretical distribution of the proper motions, in relation to position angle, is derived on the ellipsoidal theory when all proper motions below a definite value are excluded; Dyson's proposed density law

$$D(r) = \frac{A}{r} e^{-h^2k^2r^2}$$

is used in the analysis.

(*b*) A relation between $R(\mu)$—the total number of stars in the region with proper motions of μ and over—and N is obtained in terms of the ellipsoidal constants.

(*c*) A sample region has been analysed according to both the ellipsoidal and two-streams theories and the differences between the distributions $y = R(\mu)$, as derived from the two theories, are found to be almost negligible; thus in this respect the larger proper motions give no clue as to which theory is best represented by the observations.

(*d*) A general formula expressing the ratio of the axes of the velocity-ellipse (ellipsoidal theory) is derived in terms of the constants of the two-streams theory.

(*e*) A notable difference in the respective frequency curves (with respect to position angle) of the larger proper motions on the ellipsoidal and two-streams theory is indicated.

§ 2. It is well known that the observed systematic motions of the stars can generally be represented equally well in terms of the two-streams theory and of Schwarzschild's ellipsoidal hypothesis,* despite the differences in the mathematical functions by which the theories are expressed. When the proper motions provide the observed data it is essential, in the absence of an ascertained law respecting the distribution of the stars in space, to omit no star, however small its proper motion may be, from the statistics, for it is only when all proper motions are included that the theoretical characteristics of the systematic motions can be derived on the basis of either hypothesis. Although the frequency distributions given by the two-streams theory and the ellipsoidal theory are very similar, it does not necessarily follow that the two hypotheses will represent equally well the observed distributions of proper motions when only the proper motions greater than a particular value are treated statistically. In this instance, the density law according to which the stars are distributed in space must be known. In the preceding paper, the effect of introducing Dyson's density law into the analysis of the proper motions according to the two-streams theory is investigated; in the present paper, the ellipsoidal hypothesis will be considered.

* *Göttingen Nachrichten*, 1907, p. 614; A. S. Eddington, *Stellar Movements.*

§ 3. Consider a small region of the sky and the components (u, v) of a star's linear velocity projected on the tangent plane at the centre of the region. Schwarzschild's hypothesis is, that of the total number N of stars in the region, the number

$$\frac{Ngh}{\pi}e^{-g^2u^2-h^2v^2}dudv$$

will have linear velocities between (u, v) and $(u + du, v + dv)$. The axis of u is the axis of greatest mobility, g being less than h, and it points in a definite direction depending on the region concerned. If S is the solid angle subtended at the centre of the celestial sphere by the region, the volume of a conical element at a distance r is Sr^2dr and the number of stars in the volume is $Sr^2D(r)dr$ where $D(r)$ is the number of stars in unit volume at a distance r. Hence the number of stars in the element of volume with linear velocities between (u, v) and $(u + du, v + dv)$ is

$$\frac{Sgh}{\pi}r^2D(r)e^{-g^2u^2-h^2v^2}dudvdr.$$

Let U, V be the components of the parallactic motion for the region and w the resultant linear velocity of a star, making an angle θ with the u axis. Then

$$u = w\cos\theta - U, \qquad v = w\sin\theta - V,$$

and

$$dudv = wdwd\theta.$$

Then the number of stars, at a distance r, moving between position angles θ and $\theta + d\theta$ (referred to the u axis) is

$$\frac{Sgh}{\pi}r^2D(r)e^{-g^2(w\cos\theta-U)^2-h^2(w\sin\theta-V)^2}wdwd\theta dr.$$

Also $w = r\mu$ where μ is the proper motion.

Let $M(\mu, \theta)d\mu d\theta$ denote the number of stars with proper motions between μ and $\mu + d\mu$ and moving between position angles θ and $\theta + d\theta$, then

$$M(\mu, \theta)d\mu d\theta = \frac{Sgh}{\pi}\mu d\mu d\theta\int_0^\infty r^4D(r)e^{-g^2(r\mu\cos\theta-U)^2-h^2(r\mu\sin\theta-V)^2}dr \qquad (1)$$

Dyson's law is

$$D(r) = \frac{A}{r}e^{-h^2k^2r^2} \qquad . \quad . \quad . \quad . \qquad (2)$$

and we can write

$$M(\mu, \theta) = \frac{ASgh\mu}{\pi}\int_0^\infty r^3e^{-(a^2r^2-2br+c^2)}dr$$

where

$$\left.\begin{aligned} a^2 &= h^2k^2 + \mu^2(g^2\cos^2\theta + h^2\sin^2\theta) \\ b &= \mu(g^2U\cos\theta + h^2V\sin\theta) \\ c^2 &= g^2U^2 + h^2V^2 \end{aligned}\right\} \quad . \quad . \qquad (3)$$

Write

$$\frac{b}{a} = y \quad \text{and} \quad ar = x + y.$$

Then

$$\mathrm{M}(\mu, \theta) = \frac{\mathrm{ASg}h\mu}{\pi a^4} e^{y^2 - c^2} \int_{-y}^{\infty} (x + y)^3 e^{-x^2} dx.$$

If we put

$$\mathrm{I}(y) = \int_{-y}^{\infty} e^{-x^2} dx,$$

we obtain

$$\mathrm{M}(\mu, \theta) = \frac{\mathrm{ASg}h\mu}{\pi a^4} e^{y^2 - c^2} [(y^3 + \tfrac{3}{2}y)\mathrm{I}(y) + \tfrac{1}{2}(y^2 + 1)e^{-y^2}].$$

Now

$$y^2 = \frac{b^2}{a^2} = \frac{\mu^2(g^2\mathrm{U}\cos\theta + h^2\mathrm{V}\sin\theta)^2}{h^2k^2 + \mu^2(g^2\cos^2\theta + h^2\sin^2\theta)}.$$

Put

$$\left.\begin{aligned} \alpha^2 &= g^2\cos^2\theta + h^2\sin^2\theta \\ \beta &= g^2\mathrm{U}\cos\theta + h^2\mathrm{V}\sin\theta \end{aligned}\right\}$$

Then

$$y^2 = \frac{\beta^2\mu^2}{h^2k^2 + \alpha^2\mu^2} \qquad \ldots \quad (4)$$

from which we obtain

$$y dy = \frac{\beta^2 h^2 k^2}{a^4} \mu d\mu.$$

Now let $\mathrm{N}(\mu, \theta)d\theta$ denote the number of proper motions of μ and over between position angles θ and $\theta + d\theta$. Then

$$\mathrm{N}(\mu, \theta) = \int_{\mu}^{\infty} \mathrm{M}(\mu, \theta) d\mu,$$

so that

$$\mathrm{N}(\mu, \theta) = \frac{\mathrm{ASg}he^{-c^2}}{\pi\beta^2h^2k^2} \int_{y}^{y_1} [(y^3 + \tfrac{3}{2}y)\mathrm{I}(y)e^{y^2} + \tfrac{1}{2}(y^2 + 1)] y dy \ . \qquad (5)$$

where $y_1 = {}^{\beta}/_{a}$, the value of y when $\mu = \infty$. The integral on the right of (5) has already been evaluated,* and we have

$$\mathrm{N}(\mu, \theta) = \frac{\mathrm{ASg}h}{2\pi\beta^2h^2k^2} e^{-c^2} [y^3 e^{y^2}\mathrm{I}(y) + \tfrac{1}{2}y^2]_{y}^{y_1} \qquad \ldots \quad (6)$$

Using Eddington's function $f(y)$ which is given by

$$\frac{\sqrt{\pi}}{2} f(y) = y e^{y^2}\mathrm{I}(y) + \tfrac{1}{2} \qquad \ldots \quad (7)$$

we have

$$\mathrm{N}(\mu, \theta) = \frac{\mathrm{ASg}he^{-c^2}}{4\sqrt{\pi}h^2k^2\alpha^2} \left[f\left(\frac{\beta}{\alpha}\right) - \frac{\alpha^2\mu^2}{h^2k^2 + \alpha^2\mu^2} f(y) \right] \qquad \ldots \quad (8)$$

---

* *M.N.*, **88**, 572.

If N is the total number of stars in the region,

$$N = \int_0^\infty SD(r)r^2dr = AS\int_0^\infty re^{-h^2k^2r^2}dr,$$

so that

$$AS = 2h^2k^2N \quad . \quad . \quad . \quad . \quad (9)$$

Hence

$$N(\mu, \theta) = \frac{Ngh}{2\sqrt{\pi}\alpha^2}e^{-c^2}\left[f\left(\frac{\beta}{\alpha}\right) - \frac{\alpha^2\mu^2}{h^2k^2 + \alpha^2\mu^2}f(y)\right] \quad . \quad (10)$$

This equation gives us the distribution of proper motions (of μ and over) in position angle according to the ellipsoidal hypothesis combined with Dyson's density law. As in the two-streams theory, the frequency curves derived from (10) may be expected to show a well-marked dependence on the value of the constant k in the density function and on the limiting value of μ.*

§ 4. The form of (10) in the limiting case when $\frac{k}{\mu} \to 0$ is of special interest. Regarding $\frac{k}{\mu}$ as small, we have from (4)

$$y = \frac{\beta}{\alpha} - \tfrac{1}{2}\frac{\beta h^2k^2}{\mu^2\alpha^3}.$$

Hence

$$f(y) = f\left(\frac{\beta}{\alpha}\right) - \tfrac{1}{2}\frac{\beta h^2k^2}{\mu^2\alpha^3}\left(\frac{\partial f}{\partial y}\right)_{y=\beta/\alpha}$$

and consequently,

$$f\left(\frac{\beta}{\alpha}\right) - \frac{\alpha^2\mu^2}{h^2k^2 + \alpha^2\mu^2}f(y) = \frac{h^2k^2}{\mu^2\alpha^2}\left[f\left(\frac{\beta}{\alpha}\right) + \tfrac{1}{2}\frac{\beta}{\alpha}\left(\frac{\partial f}{\partial y}\right)_{y=\frac{\beta}{\alpha}}\right]. \quad (11)$$

Now from (7),

$$y\frac{\partial f}{\partial y} = (2y^2 + 1)f(y) - \frac{1}{\sqrt{\pi}},$$

and the expression inside the brackets on the right-hand side of (11) becomes

$$\tfrac{1}{2}\left[\left\{3 + 2\left(\frac{\beta}{\alpha}\right)^2\right\}f\left(\frac{\beta}{\alpha}\right) - \frac{1}{\sqrt{\pi}}\right].$$

From (8),

$$N(\mu, \theta) = \frac{ASghe^{-c^2}}{8\sqrt{\pi}\mu^2\alpha^4}\left[\left\{3 + 2\left(\frac{\beta}{\alpha}\right)^2\right\}f\left(\frac{\beta}{\alpha}\right) - \frac{1}{\sqrt{\pi}}\right] \quad . \quad (12)$$

It is evident that (12) gives the theoretical distribution of proper motions (of μ and over) in position angle in the limiting case when $k = 0$. Also it follows from the analysis that (12) is the approximate result when $\frac{k}{\mu}$ is small, that is, for a definite non-zero value of k and a large value of μ, the limiting proper motion.

* For example, see fig. 3, *M.N.*, **88**, 579.

§ 5. We shall now obtain the formula giving the total number of stars in the region with proper motions of μ and over. Denote this number by $R(\mu)$. Then

$$R(\mu) = \int_0^{2\pi} N(\mu, \theta) d\theta,$$

in which $N(\mu, \theta)$ is given by equation (10); y, α, and β are all functions of θ. Hence

$$R(\mu) = \frac{Ngh}{2\sqrt{\pi}} e^{-c^2} \left[\int_0^{2\pi} \frac{1}{\alpha^2} f\left(\frac{\beta}{\alpha}\right) d\theta - \int_0^{2\pi} \frac{\mu^2}{h^2k^2 + \alpha^2\mu^2} f(y) d\theta \right] \quad (13)$$

Let $\rho d\theta$ denote the total number of stars in the region with proper motions between 0 and ∞ and between position angles θ and $\theta + d\theta$; then putting $\mu = 0$ in (10) we have

$$\rho d\theta = \frac{Ngh}{2\sqrt{\pi}} e^{-c^2} \frac{1}{\alpha^2} f\left(\frac{\beta}{\alpha}\right) d\theta.$$

Now

$$\int_0^{2\pi} \rho d\theta = N;$$

hence

$$\frac{Ngh e^{-c^2}}{2\sqrt{\pi}} \int_0^{2\pi} \frac{1}{\alpha^2} f\left(\frac{\beta}{\alpha}\right) d\theta = N,$$

so that

$$\int_0^{2\pi} \frac{1}{\alpha^2} f\left(\frac{\beta}{\alpha}\right) d\theta = \frac{2\sqrt{\pi} e^{c^2}}{gh} \quad . \quad . \quad . \quad (14)$$

We have now to find the value of $\int_0^{2\pi} \frac{\mu^2}{h^2k^2 + \alpha^2\mu^2} f(y) d\theta$ in (13) where y, α, and β are defined by (4). We have

$$\begin{aligned} \frac{h^2k^2 + \alpha^2\mu^2}{\mu^2} &= \frac{h^2k^2}{\mu^2} + \alpha^2 \\ &= \frac{h^2k^2}{\mu^2} + g^2 + (h^2 - g^2) \sin^2\theta \\ &= G^2 + (H^2 - G^2) \sin^2\theta \\ &= G^2 \cos^2\theta + H^2 \sin^2\theta, \end{aligned}$$

where

$$G^2 = g^2 + \frac{h^2k^2}{\mu^2}$$

and

$$H^2 = h^2 + \frac{h^2k^2}{\mu^2} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Write

$$\frac{h^2k^2 + \alpha^2\mu^2}{\mu^2} = A^2;$$

then A^2 and α^2 are analogous functions of θ.

Also
$$y^2 = \frac{\beta^2}{\alpha^2 + \dfrac{h^2k^2}{\mu^2}} = \frac{\beta^2}{A^2},$$
in which
$$\beta = g^2U\cos\theta + h^2V\sin\theta.$$
Write
$$\beta = G^2U_1\cos\theta + H^2V_1\sin\theta,$$
$$\equiv B,$$
where
$$G^2U_1 = g^2U \quad \text{and} \quad H^2V_1 = h^2V.$$
We thus obtain
$$\left.\begin{aligned} G^2{U_1}^2 &= \frac{g^4U^2}{g^2 + \dfrac{h^2k^2}{\mu^2}} \\ \text{and} \qquad H^2{V_1}^2 &= \frac{h^4V^2}{h^2 + \dfrac{h^2k^2}{\mu^2}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (16)$$

We can now write down the value of the second integral in (13) and find by using (14)
$$\int_0^{2\pi} \frac{\mu^2}{h^2k^2 + \alpha^2\mu^2} f(y)d\theta = \int_0^{2\pi} \frac{1}{A^2} f\left(\frac{B}{A}\right)d\theta$$
$$= \frac{2\sqrt{\pi}}{GH} e^{C^2} \quad . \quad . \quad . \quad (17)$$
Here C^2 is given by
$$C^2 = G^2{U_1}^2 + H^2{V_1}^2$$
$$= \frac{g^2U^2}{1 + \dfrac{h^2k^2}{g^2\mu^2}} + \frac{h^2V^2}{1 + \dfrac{k^2}{\mu^2}}$$
by means of (16).

We then have
$$R(\mu) = N\left[1 - \frac{gh}{GH} e^{C^2 - c^2}\right] \quad . \quad . \quad . \quad (18)$$

Let ϵ denote $\frac{g}{h}$, the ratio of the axes in the velocity-ellipse. We then obtain, after some reduction,
$$R(\mu) = N\left[1 - \frac{e^{-\left\{\frac{g^2U^2}{1 + \frac{\epsilon^2\mu^2}{k^2}} + \frac{h^2V^2}{1 + \frac{\mu^2}{k^2}}\right\}}}{\sqrt{\left(1 + \dfrac{k^2}{\epsilon^2\mu^2}\right)\left(1 + \dfrac{k^2}{\mu^2}\right)}}\right] \quad . \quad (19)$$

If the ellipsoidal elements g, h, U, V, and the density constant k are known, this expression gives the total number of stars in the region with proper motions of μ and over.

The corresponding formula * for the two-streams theory is, when the total number of stars in the region are equally divided between the two streams,

$$R(\mu) = \frac{N}{2}[1 - \eta^2 e^{-h^2V_1^2(1-\eta^2)}] + \frac{N}{2}[1 - \eta^2 e^{-h^2V_2^2(1-\eta^2)}] \quad . \quad (20)$$

in which hV_1, hV_2 are the drift-velocities and η is defined in terms of k and the limiting proper motion μ by

$$\eta = \frac{\mu}{(\mu^2 + k^2)^{\frac{1}{2}}}.$$

It is thus possible, by means of (19) and (20), to test whether the two-streams theory or the ellipsoidal theory gives the better accordance with the observed proper motions of μ and over.

Again, the formula (10) gives the theoretical frequency curve for the proper motions of μ and over, and this may quite possibly show distinct differences from the corresponding frequency curve derived on the two-streams theory.

§ 6. We shall now consider a special region in the sky of which the centre is $0^h\ 15^m, +50°$. This position corresponds to the centre of Group I. of the Cambridge photographic regions and we have at once the drift constants for this part of the sky; they are

Drift I.: drift-velocity $hV_1 = 1{\cdot}5$: position angle 105°
Drift II.: „ „ $hV_2 = 0{\cdot}8$: „ „ 190°,

and the stars of the region are equally distributed between the two drifts. From the theoretical frequency curve based on these values of the drift constants, the constants of the ellipsoidal hypothesis were obtained, following the procedure in Professor Eddington's *Stellar Movements* (p. 130). The analysis gave the following results :—

$$\epsilon \equiv \frac{g}{h} = 0{\cdot}63,$$

$$gU = 0{\cdot}32,$$

$$hV = 0{\cdot}66,$$

from which

$$hU = 0{\cdot}51,$$

and

$$h(U^2 + V^2)^{\frac{1}{2}} = 0{\cdot}83,$$

and

$$\tan^{-1}\frac{V}{U} = 52\tfrac{1}{2}°.$$

Also the direction of greatest mobility, that is the u-axis in the theory, is in position angle 77°. Thus the parallactic motion, expressed in terms of the unit $1/h$, is 0·83 in position angle $77° + 52\frac{1}{2}°$ or $129\frac{1}{2}°$. We can compare these results with the corresponding results from the

* *M.N.*, **88**, 580.

two-streams theory. The solar motion from the analysis of the Cambridge proper motions was found to be 0·88 ; from the derived position of the solar ant-apex, the angular distance of the region which we are considering from the former is 120° and the position angle of the solar ant-apex from the centre of the region is 121°. The parallactic motion is thus 0·88 sin 120° or 0·76 in position angle 121°. The accordance between the results for the two theories may be regarded as a check on the calculations.

We can now insert the ellipsoidal constants and the two-drifts constants in (19) and (20) and for given values of μ and k obtain a comparison between the implications of the two theories. In a previous paper * it was found that the density function, with values of k between 0″·0 and 2″·0 depending on galactic latitude, was not inconsistent with the observations. The galactic latitude of our region is − 11°, so that the value of k is probably about 0″·5 ; however, the calculations have been made for the following values of k: 2″·0, 1″·0, 0″·5, 0″·0. The values of μ considered are 2″, 4″, 6″, and 8″. The results are summarised in the following table in which E. denotes the " ellipsoidal theory " [equation (19)] and T.S. denotes the " two-streams theory " [equation (20)]. When $k = 0$, both (19) and (20) reduce to the form

$$R(\mu) = \frac{C}{\mu^2}.$$

Also, for the purposes of comparison, $R(\mu)$ is taken to be 100 when $\mu = 2''$ per century.

TABLE.

Values of $R(\mu)$.

μ.	$k=2''\cdot0$.		$k=1''\cdot0$.		$k=0''\cdot5$.		$k=0''\cdot0$.	
	E.	T.S.	E.	T.S.	E.	T.S.	E.	T.S.
″								
2	100	100	100	100	100	100	100	100
4	54·4	53·6	34·7	34·3	27·7	27·3	25·0	25·0
6	30·2	29·2	17·0	16·3	12·6	12·6	11·1	11·1
8	19·2	18·4	9·8	9·4	7·1	7·1	6·2	6·2

Within the range of μ considered here the two formulæ (19) and (20) lead to very much the same numerical values, and it is hardly likely that the values of $R(\mu)$ obtained from actual counts of proper motions would be able to differentiate between the two theories. If we continue the calculations for values of μ greater than 8″, the agreement between the two columns E. and T.S. would become more perfect with increase of μ, as can be easily seen from (19) and (20) when $\frac{k}{\mu}$ decreases in value. Moreover, if we had restricted our calculations to rather large values of μ, say 20″ and over, the agreement between the two sets of values of $R(\mu)$ would be even closer than is indicated in the

* *M.N.*, **88**, 567.

preceding table. It transpires from this discussion that the frequency distribution of proper motions taken *according to their magnitudes* is practically identical in the two theories; in other words, the curve $y = R(\mu)$, when computed from (19), is almost identical with that computed from (20). The restriction of proper motions by the method of this section offers no clue as to which theory best represents any given set of observations.

§ 7. The conclusions of the preceding section enable us to derive an important relation involving the ratio ϵ (the ratio of the axes of the velocity-ellipse), the drift-velocities hV_1, hV_2 in the region concerned, and the solar velocity. If the systematic motions have been analysed on the basis of the two-streams theory, it is then a simple matter to derive the ratio of the axes in the velocity-ellipse. For $k = 0$, the ratio of $R(\mu)$ to N is the same for both theories—as the last two lines in the preceding table show. Also $R(\mu) : N$ is practically the same for both theories when the values of μ are large. Thus when $\frac{k}{\mu}$ is small, the right-hand sides of (19) and (20) may be taken to be equal.

Taking (19) first, we have when $\frac{k}{\mu}$ is small:

$$\left[\left(1 + \frac{k^2}{\epsilon^2\mu^2}\right)\left(1 + \frac{k^2}{\mu^2}\right)\right]^{-\frac{1}{2}} = 1 - \frac{k^2}{2\mu^2}\left(1 + \frac{1}{\epsilon^2}\right)$$

$$\frac{g^2U^2}{1 + \frac{\epsilon^2\mu^2}{k^2}} + \frac{h^2V^2}{1 + \frac{\mu^2}{k^2}} = \frac{k^2}{\mu^2}\left\{\frac{g^2U^2}{\epsilon^2} + h^2V^2\right\}$$

$$= \frac{k^2}{\mu^2}h^2(U^2 + V^2).$$

Hence,

$$R(\mu) = N\frac{k^2}{\mu^2}\left[h^2(U^2 + V^2) + \tfrac{1}{2}\left(1 + \frac{1}{\epsilon^2}\right)\right] \quad . \quad . \quad (21)$$

Consider now formula (20). If $\frac{k}{\mu}$ is small, η is approximately unity. Write $\alpha = 1 - \eta$. Then

$$\alpha = 1 - \frac{\mu}{(\mu^2 + k^2)^{1/2}}$$

$$= \tfrac{1}{2}\frac{k^2}{\mu^2},$$

and after some reduction

$$1 - \eta^2 e^{-h^2V_1^2(1-\eta^2)} = \frac{k^2}{\mu^2}(1 + h^2V_1^2).$$

We thus find from (20), the total number of stars in the region being equally distributed between the two streams, that

$$R(\mu) = N\frac{k^2}{\mu^2}[1 + \tfrac{1}{2}h^2V_1^2 + \tfrac{1}{2}h^2V_2^2] \quad . \quad . \quad (22)$$

Equating (21) and (22) we have

$$\frac{1}{\epsilon^2} \equiv \frac{h^2}{g^2} = 1 + h^2V_1{}^2 + h^2V_2{}^2 - 2h^2(U^2 + V^2).$$

Now $h(U^2 + V^2)^{\frac{1}{2}}$ is the parallactic motion for the region concerned, so that if hW denotes the solar motion and λ the angular distance of the region from the solar ant-apex, we have

$$h^2(U^2 + V^2) = h^2W^2 \sin^2 \lambda.$$

Hence

$$\frac{1}{\epsilon^2} = 1 + h^2V_1{}^2 + h^2V_2{}^2 - 2h^2W^2 \sin^2 \lambda \quad . \quad . \quad (23)$$

This equation enables us to calculate ϵ, the ratio of the axes of the velocity-ellipse, for any given region provided the drift-velocities for that region and the solar velocity are known.

For the region considered in § 6, $hV_1 = 1{\cdot}5$ and $hV_2 = 0{\cdot}8$. Also $hW = 0{\cdot}88$ and $\lambda = 120°$—from the complete analysis of the Cambridge proper motions.* Inserting these values in (23) we obtain

$$\epsilon \equiv \frac{g}{h} = 0{\cdot}60,$$

in good agreement with the value 0·63 found in § 6 from the analysis of the frequency curve.

Provided that the stars—on the two-streams theory—are equally divided between the two streams, as several investigations of the proper motions of faint stars show, the formula (23) is a general one, applicable to any region of the sky.

§ 8. Let us return to equation (12) in § 4. This formula gives the number N(μ, θ) of stars moving between θ and $\theta + d\theta$ with proper motions of μ and over in the limiting case when $k = 0$ and also when μ is large. It is comparatively easy to compare the frequency curve of proper motions, for any given limiting value of μ, given by (12) with the corresponding frequency curve derived from the two-streams theory. The calculations have been made, as regards the application of (12), with the ellipsoidal constants given in § 6. In (12), α and β are functions of θ, θ being measured from the u axis—the axis of maximum mobility—which was found to have a true position angle (with respect to the meridian through the centre of the region) of 77°.

For a single drift of stars, N_1 in number, extending up to a distance R,

$$N(\mu, \theta) = \frac{ASe^{-h^2V^2}}{8\sqrt{\pi}h^2\mu^2}\left[(3 + 2\tau^2)f(\tau) - \frac{1}{\sqrt{\pi}}\right]$$

where $$\tau = hV \cos(\theta - \theta_1),$$

and $$N_1 = \tfrac{1}{2}ASR^2,$$

and θ_1 is the direction in which the drift is pointing.† For the two drifts of the region under consideration, we shall take the numbers N_1 and N_2 equal.

Fig. 1 shows the results of the computations. To make the com-

* *M.N.*, **87**, 137. † See *M.N.*, **88**, 574.

parison of the two theories possible, the total number of stars with proper motions of μ and over has been taken to be 1000—the areas under the curves are thus equal. Actually the limiting value of μ has been chosen to be 2″. For this particular region the differences between the two frequency curves are rather striking. The ellipsoidal frequency curve smooths out the maximum near position angle 190°, which appears conspicuously in the frequency curve derived from the two-streams theory.

Fig. 1 is of course only one illustration of the differences between

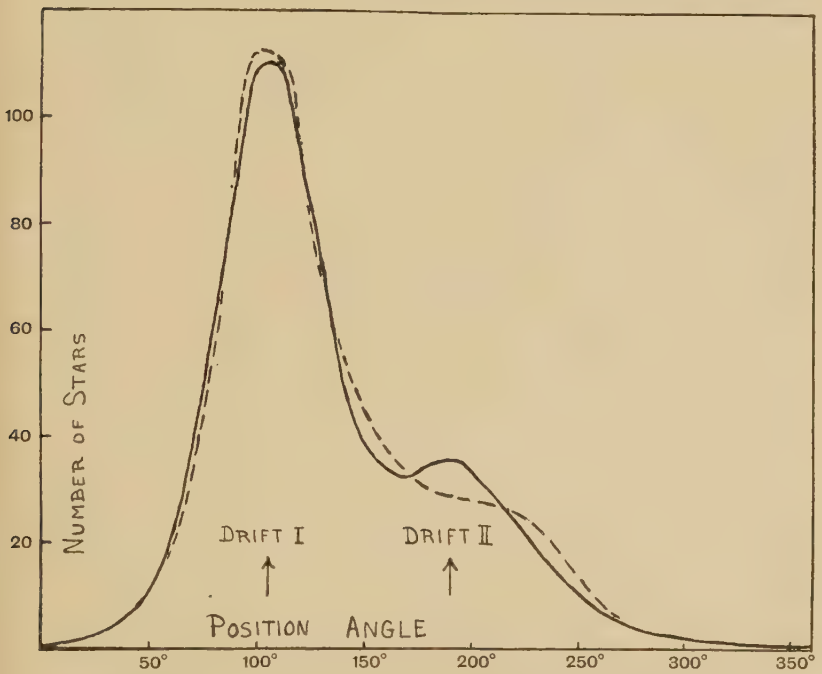


FIG. 1.—Full Curve—Two-streams Theory; Broken Curve—Ellipsoidal Theory.

the two theories—a more general one would be based on formula (10) and its counterpart in the two-streams theory. In the most usual case when every proper motion, however small it may be, is brought into the statistics, the differences between the two theories are inconspicuous; it would seem from fig. 1 that such differences are magnified considerably when the smaller proper motions are excluded and are probably large enough to provide a criterion by which the observed proper motions may be called upon to decide which of the two theories —apart from the physical ideas connected with them—best accords with the observational data.

Observatory, Cambridge:
1928 November 5.

Giant and Dwarf Stars of Type Ko. By Sir Frank Dyson.

1. The determinations of stellar parallax at Greenwich being confined to one part of the sky contain a large proportion of stars of moderate proper motion. In consideration of the best way of extending the programme of observations it seemed that a useful field lay in the determination of the Giant and Dwarf Ko stars of different magnitudes. In Russell's diagram there is a concentration of Absolute Magnitude about $0^{m}\cdot0$ and another about $5^{m}\cdot5$, and intermediate magnitudes are gradually being put in. It is quite clear there is no sharp division as with the K5 stars, but it is of interest to make out what proportion of Ko stars, say of observed absolute magnitude $3^{m}\cdot0$, are to be classed as "Giants" and what proportion as "Dwarfs."

Schlesinger's Catalogue of Trigonometrical Parallaxes is complete for North declination for type Ko (with a few additions since published) as far as the stars of Visual Magnitude $4^{m}\cdot5$. From $4^{m}\cdot5$ to $5^{m}\cdot0$ the parallaxes of 31 stars have been determined. These include 5 stars of proper motion greater than $''\cdot20$, and 26 with proper motion less than this. There are 22 stars, all of proper motion less than $''\cdot20$ whose parallaxes have not been determined. For magnitudes between $5^{m}\cdot0$ and $5^{m}\cdot5$, the parallaxes of about a quarter of the stars have been found, generally speaking stars of large proper motions. The stars fainter than $5^{m}\cdot5$ whose parallaxes have been found have been selected largely though not entirely on account of large proper motion.

In the following table the stars between magnitudes $4^{m}\cdot0$ and $4^{m}\cdot5$ are given. The approximate R.A. is given in column 1 for identification; column 2 gives the Harvard magnitude; column 3, the proper motion from Schlesinger's Catalogue; column 4, the parallax. In column 5 the parallax is corrected by Eddington's method (*M.N.*, 1926, p. 687) according to the following table, in which the parallaxes are expressed in a unit of $''\cdot001$.

−32 to −12	taken as	5	11 to 14 unaltered
−11 „ − 7	„	6	From 15 to 17 add −1
− 6 „ 0	„	7	„ 18 „ 22 „ −2
0 „ 4	„	8	„ 23 „ 27 „ −3
5 „ 7	„	9	„ 28 „ 100 „ −4
8 „ 9	„	10	
10	„	11	

In column 6 is given $m + 5 \log \mu + 5 = H$.
In column 7 is given $m + 5 \log \pi + 5 = M$.
The effect of the corrections where they are negative is to reduce

the absolute magnitudes by $0^m.1$, $0^m.2$, or $0^m.3$. For the parallaxes $> ''.100$ they have no effect.

Parallaxes of Ko *Stars*, $4^m.0$–$4^m.5$.

R.A. h	R.A. m	m.	μ ″·001.	π ″·001.	π (Corr^d.) ″·001.	H.	M.
0	42	4·30	129	34	30	4·9	1·7
0	58	4·45	81	25	22	4·0	1·2
3	2	4·00	242	29	25	5·9	1·0
4	22	4·04	111	31	27	4·3	1·2
4	50	4·28	103	19	17	4·3	0·4
5	31	4·39	321	27	24	6·9	1·3
5	44	4·18	7	14	14	−1·6	−0·1
6	9	4·45	273	14	14	6·6	0·2
7	4	4·48	51	5	9	3·0	−0·7
7	37	4·26	246	20	18	6·2	0·5
8	39	4·17	238	16	15	6·0	0·0
9	47	4·10	225	19	17	5·9	0·3
10	22	4·41	158	11	11	5·5	−0·3
13	44	4·28	106	−18	5	4·4	−2·2
17	21	4·44	4	−10	6	−2·6	−1·7
17	26	4·48	24	16	15	−1·3	0·4
18	0	4·07	1131	192	192	9·3	5·5
18	16	4·34	38	3	8	2·2	−1·1
18	55	4·21	98	21	19	4·2	0·6
19	12	4·46	8	3	8	−1·0	−1·0
19	36	4·45	39	12	12	2·4	−0·2
19	52	4·03	51	12	12	2·6	−0·6
20	41	4·34	24	13	13	1·2	−0·1
21	30	4·22	100	6	9	4·2	−1·0

Examination of the last column naturally divides the stars into 23 "Giants" with absolute magnitudes $-2^m.2$ to $+1^m.7$ and 1 "Dwarf" of magnitude $5^m.5$. In the 121 stars brighter than $5^m.5$ whose parallaxes are given there is a range from $-2^m.5$ to $3^m.2$ and then a break to 4 stars of magnitudes $5^m.0$, $5^m.1$, $5^m.5$, and $6^m.2$.

2. Taking first all parallaxes $> ''.020$. To $m = 4^m.5$, the list of stars is complete and gives

2 stars with M between	$-1^m.0$	and	$0^m.0$
13 „ „ „	$0^m.0$	„	$1^m.0$

With $m > 4^m.5$ no stars with $\pi > ''.020$ will come into these groups.

To $m = 4^m.5$ there are 11 stars with M between $1^m.0$ and $2^m.0$.

Between the limits $m = 4^{m}\cdot5$ and $5^{m}\cdot0$ there are 9 stars with parallax determined $> ''\cdot020$, giving

4 stars	with M	$1^{m}\cdot0$	to $2^{m}\cdot0$
2 ,,	,,	$2^{m}\cdot0$	,, $3^{m}\cdot0$
2 ,,	,,	$3^{m}\cdot0$	,, $4^{m}\cdot0$
1 star	,,	$6^{m}\cdot1$	

The number of undetermined parallaxes $> ''\cdot020$ which will give values of M between $1^{m}\cdot0$ and $2^{m}\cdot0$ may be estimated at 4.

Between the limits $4^{m}\cdot5$ and $5^{m}\cdot0$ there are 7 stars with $\pi > ''\cdot020$, giving the values of M $1^{m}\cdot4$, $1^{m}\cdot5$, $2^{m}\cdot0$, $2^{m}\cdot5$, $2^{m}\cdot6$, $3^{m}\cdot1$, and $5^{m}\cdot2$. As the parallaxes of one-quarter of the stars have been determined, the total number with M between $1^{m}\cdot0$ and $2^{m}\cdot0$ in this group may be assumed to be 8.

Thus with $\pi > ''\cdot020$ there are 11 + 8 + 8 or 27 stars with M between $1^{m}\cdot0$ and $2^{m}\cdot0$.

If parallaxes $> ''\cdot030$ had been taken for which only stars are needed to $4^{m}\cdot6$, 6 stars would have been found, and for $\pi > ''\cdot040$, 3 stars: leading to $\frac{27}{8} \times 6$ and 8×3, or 20 and 24 as the numbers in the range given by $\pi > ''\cdot020$. From these figures we may take it that in North declination there are within 50 parsecs of the Sun :—

2 Ko stars	of absolute magnitude	$-1^{m}\cdot0$	to $0^{m}\cdot0$
13	,,	$0^{m}\cdot0$	,, $1^{m}\cdot0$
24	,,	$1^{m}\cdot0$	,, $2^{m}\cdot0$

For stars of absolute magnitude fainter than $2^{m}\cdot0$ Schlesinger's Catalogue is far from complete for so large a distance as 50 parsecs, but gradually tends to completion as the distance is diminished to 10 parsecs for stars as faint as absolute magnitude $8^{m}\cdot5$, as most of the Ko stars in the Draper Catalogue with large proper motions have had their parallaxes determined.

Number of Stars with M *between Certain Limits.*

M. / π	$-1\cdot0$–$0\cdot0$.	$0\cdot0$–$1\cdot0$.	$1\cdot0$–$2\cdot0$.	$2\cdot0$–$3\cdot0$.	$3\cdot0$–$4\cdot0$.	$4\cdot0$–$5\cdot0$.	$5\cdot0$–$6\cdot0$.	$6\cdot0$–$7\cdot0$.	$7\cdot0$–$8\cdot0$.
$> ''\cdot020$	2	13	16	9	14	15	40	28	5
$> \cdot030$	1	4	6	3	3	5	30	24	4
$> \cdot040$	1	1	3	1	3	5	22	18	3
$> \cdot050$	1	1	0	0	2	3	20	13	3
$> \cdot060$	1	1	0	0	2	3	12	9	2
$> \cdot070$	1	1	0	0	1	3	11	6	1
$> \cdot080$	0	1	0	0	0	2	10	6	1
$> \cdot090$	0	1	0	0	0	2	7	5	0
$> \cdot100$	0	1	0	0	0	2	5	5	0

Multiplying by suitable factors, the numbers of stars of different magnitudes in the hemisphere of 50 parsecs radius are as follows, where the nine lines refer to parallaxes greater than certain amounts as in the previous table. Underneath are repeated the more accurate values for absolute magnitudes brighter than − 1·0. For the absolutely bright stars the smaller parallaxes give the more accurate results; for fainter stars the larger parallaxes.

−1·0–0·0.	0·0–1·0.	1·0–2·0.	2·0–3·0.	3·0–4·0.	4·0–5·0.	5·0–6·0.	6·0–7·0.	7·0–8·0
2	13	16	9	14	15	40	28	5
3	13	20	10	10	17	101	81	13
8	8	24	8	24	40	176	144	24
				32	47	312	203	47
				54	81	324	243	54
				43	130	472	257	43
					128	640	384	64
					182	638	455	
					250	625	625	
2	13	24						

3. These results can be usefully supplemented by examination of the parallax determinations made at Greenwich, where the sky has been more completely examined for stars of moderate proper motion. The parallaxes of all Ko stars with proper motion $>''\cdot 20$ in the area 64° to the pole have been determined and a large proportion of those with proper motion between $''\cdot 10$ and $''\cdot 20$. The following table (p. 120) gives all the Ko stars brighter than $9^{m}\cdot 0$ with their proper motions and the parallaxes of those already determined. The magnitudes given are derived from the Greenwich photographic magnitudes by applying the value $-0^{m}\cdot 8$ found by Mr. Martin to be required to reduce them to the Harvard Visual Scale. The parallaxes given are those actually determined and are not corrected by the table on p. 116.

(1) Out of 6 stars from $5^{m}\cdot 0$ to $5^{m}\cdot 9$, 3 have parallaxes $>''\cdot 020$. The parallaxes (corrected) give values of M $3^{m}\cdot 3$ and $2^{m}\cdot 3$. It may be assumed that one of three stars of unknown parallax will have a parallax $>''\cdot 020$, and the value of M may be assumed to be between $2^{m}\cdot 0$ and $3^{m}\cdot 0$.

(2) The 5 stars $6^{m}\cdot 0$–$6^{m}\cdot 9$ with $\mu <''\cdot 20$ give one parallax $''\cdot 024$ ($''\cdot 021$ corrected) and give a value of M $3^{m}\cdot 4$. The chance that one of the two stars of unknown parallax has a parallax $>''\cdot 020$ is about $\frac{2}{5}$, and thus we may add $\frac{2}{5}$ of a star with M between $3^{m}\cdot 0$ and $4^{m}\cdot 0$.

(3) Out of 18 stars from $7^{m}\cdot 0$ to $7^{m}\cdot 9$ and $\mu <''\cdot 20$, 5 have parallaxes $>''\cdot 020$ (corrected) and give rise to values of M $6^{m}\cdot 6$, $3^{m}\cdot 5$, $4^{m}\cdot 7$, $4^{m}\cdot 6$, and $3^{m}\cdot 7$. The remaining 10 may be expected to give 2 stars with M $4^{m}\cdot 0$ to $5^{m}\cdot 0$, and add $\frac{1}{9}$ of a star from $3^{m}\cdot 0$ to $4^{m}\cdot 0$.

(4) Out of 17 stars $8^{m}\cdot 0$–$8^{m}\cdot 9$ and μ from $''\cdot 1$ to $''\cdot 2$, 9 in the last

column have $\pi > ''\cdot020$, and give values of M $5^{m}\cdot5$, $5^{m}\cdot2$, $5^{m}\cdot7$, $4^{m}\cdot9$, $5^{m}\cdot6$, $5^{m}\cdot9$, $5^{m}\cdot6$, $5^{m}\cdot2$, and $5^{m}\cdot2$. The remaining 9 stars may be expected to give 5 stars with M $5^{m}\cdot0$–$6^{m}\cdot0$.

Ko *Stars from* 64° *to* 90° *with Proper Motions* $>''\cdot10$.

m.	μ ″·01.	π ″·001.	m.	μ ″·01.	π ″·001.	m.	μ ″·01.	π ″·001.
4·7	180	168	7·1	33	51	8·2	35	35
5·9	22	46	7·8	40	18	8·2	48	26
3·8	17	69	7·1	60	38	8·4	51	33
3·2	14	38	7·9	16	− 2	8·4	30	15
4·0	14	40	7·8	10	16	8·5	23	14
4·9	16	− 3	7·9	19	55	8·2	10	34
5·3	17	43	7·6	11	6	8·2	14	11
5·1	14	− 8	7·5	14	− 5	8·5	12	13
5·7	15	3	7·9	11	5	8·2	19	29
5·5	18	26	7·0	10	23	8·5	11	6
5·2	10	16	7·9	13	0	8·5	10	31
5·6	17		7·8	10	0	8·4	11	23
5·9	15		7·9	15	26	8·2	16	34
5·8	10		7·9	13	14	8·8	12	30
			7·9	12	25	8·3	11	−22
6·0	54	68	7·0	18	15	8·2	12	34
6·9	10	−26	7·5	15	0	8·4	11	18
6·1	10	14	7·2	12	25	8·0	15	34
6·4	16	4	7·3	11	13	8·7	17	22
6·8	19	24	7·2	12	23	8·6	17	7
6·6	18	14	7·3	13	− 2	8·1	12	30
6·8	11	23	7·8	17	−12	8·5, 8·7	16	
6·0	17		7·3	10				
			7·3	10		8·2	14	
			7·4	14		8·1	10	
			7·0	12		8·2	10	
			7·6	13		8·5	14	
			7·2	10		8·1	15	
			7·9	13		8·7	10	
			7·0	12		8·5	10	
			7·1	14				

From known parallaxes $>''\cdot020$, the Greenwich observations give

M.	1·0–2·0.	2·0–3·0.	3·0–4·0.	4·0–5·0.	5·0–6·0.
No	4	2	5	5	16

Add to these for stars of p.m. ″·1 to ″·2 and unknown parallax :

1	1	2	5

For the stars of proper motion <″·010, the number of parallaxes >″·020 is largely a matter of guess. In Schlesinger's Catalogue there are 18 Ko stars of magnitudes $5^{m}\cdot5$ to $6^{m}\cdot5$ with proper motions between ″·10 and ″·05, and two of these have parallaxes >″·020.

In the region of the sky observed for parallax at Greenwich, the number of stars with proper motion 0 to ″·05 and ″·05 to ″·10 are as shown below :—

	$5^{m}\cdot0$–$6^{m}\cdot0$.	$6^{m}\cdot0$–$7^{m}\cdot0$.	$7^{m}\cdot0$–$8^{m}\cdot0$.	$8^{m}\cdot0$–$9^{m}\cdot0$.
μ ″·05 to ″·10	13	24	48	52
μ ·00 ,, ·05	20	66	201	347

From the numbers with p.m. ″·10 to ″·20 observed at Greenwich and ″·05 to ″·10 in Schlesinger's Catalogue, I have assumed that the number of stars with parallax >″·020 in these groups are 2, 4, 10, 15, and we may take these as the number of stars of absolute magnitude 2·0–3·0, 3·0–4·0, 4·0–5·0, and 5·0–6·0. Thus we obtain from the Greenwich stars a number with π >″·020 of absolute magnitudes as follows :—

M.	1·0–2·0.	2·0–3·0.	3·0–4·0.	4·0–5·0.	5·0–6·0.
	4	5	10	17	36

or in the hemisphere, nine times this :

36	45	90	153	324

As the limiting apparent magnitude of the stars considered is very slightly over $8^{m}\cdot5$, the number in the last group should be doubled.

The best values I can get from Schlesinger's Catalogue and the Greenwich results are :

M.	No.	M.	No.
−1·0 to 0	2	3·0 to 4·0	100
0 ,, 1·0	13	4·0 ,, 5·0	200
1·0 ,, 2·0	24	5·0 ,, 6·0	600
2·0 ,, 3·0	40		

The most uncertain of these numbers are the number from $2^{m}\cdot0$ to $3^{m}\cdot0$. The parallaxes of stars of 6th and 7th magnitude of small proper motion are required before this can be accurately determined.

4. The values of $m + 5 \log \mu + 5$ or H have been computed for the Ko stars of the Zone 64° to 72° for which proper motions have been determined by comparison of recent photographs with those taken 30 years ago. As $m + 5 \log \mu = M + 5 \log T + C$, where T is the linear cross-velocity (including the solar motion), and as large luminosities are associated with small velocities, it was thought that this tabulation might be used to throw some light on the distribution of M.

The number of stars between different limits are shown in Table III. and diagram 1, the Ao stars being added to the table for comparison.

H +	Number of Stars. Ko.	Ao.	H +	Number of Stars. Ko.	Ao.
−0·8 to −0·3	4	5	4·7 to 5·2	55	27
−0·3 ,, +0·2	5	2	5·2 ,, 5·7	45	6
+0·2 ,, +0·7	0	9	5·7 ,, 6·2	60	5
+0·7 ,, 1·2	9	12	6·2 ,, 6·7	37	1
1·2 ,, 1·7	8	14	6·7 ,, 7·2	21	
1·7 ,, 2·2	17	17	7·2 ,, 7·7	15	
2·2 ,, 2·7	17	41	8·2 ,, 8·7	13	
2·7 ,, 3·2	24	37	8·7 ,, 9·2	9	
3·2 ,, 3·7	33	51	9·2 ,, 9·7	9	
3·7 ,, 4·2	49	56	9·7 ,, 10·2	2	
4·2 ,, 4·7	57	32	10·2	7	

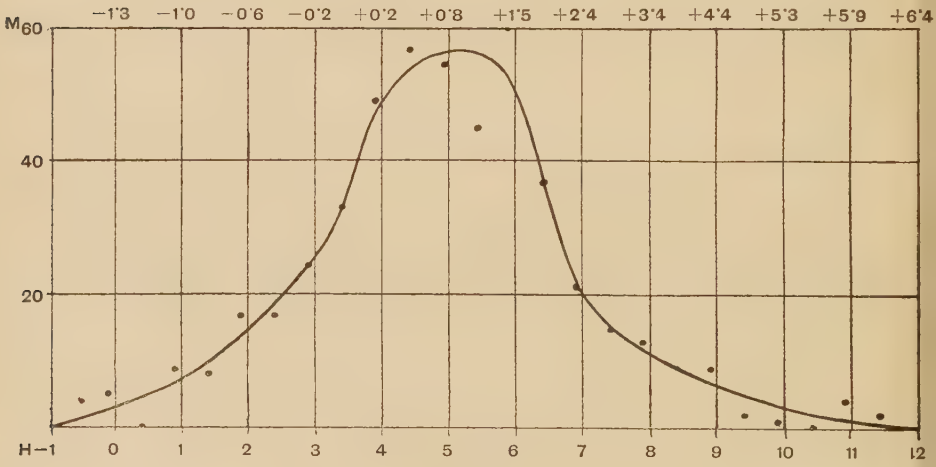


Fig. 1.

The diagram does not show the anticipated division between "giants" and "dwarfs," but shows how, with a number of variable elements, distribution tends to take the form of an error curve.

5. An attempt has been made to derive from this diagram the distribution of these stars in absolute magnitude. From the Solar Motion, omitting 5 stars of large proper motion, the mean absolute magnitude of the stars was found to be $1^{m}\cdot2$ and the corresponding mean value of H $4^{m}\cdot9$. Omitting the stars for which H was $<2^{m}\cdot9$ or $>6^{m}\cdot9$ the mean value of π was found to be $''\cdot0040$ and of M $0^{m}\cdot7$.

Next the values of M and H obtained from Schlesinger's Catalogue

and some later values were plotted against one another as shown in diagram 2. A line was drawn through these widely spreading points, and gives the following results :—

H.	M.	H.	M.
	m		m
0·0	−1·3	7·0	+2·4
1·0	−1·0	8·0	+3·4
2·0	−0·6	9·0	+4·4
3·0	−0·2	10·0	+5·3
4·0	+0·2	11·0	+5·9
5·0	+0·8	12·0	+6·4
6·0	+1·5		

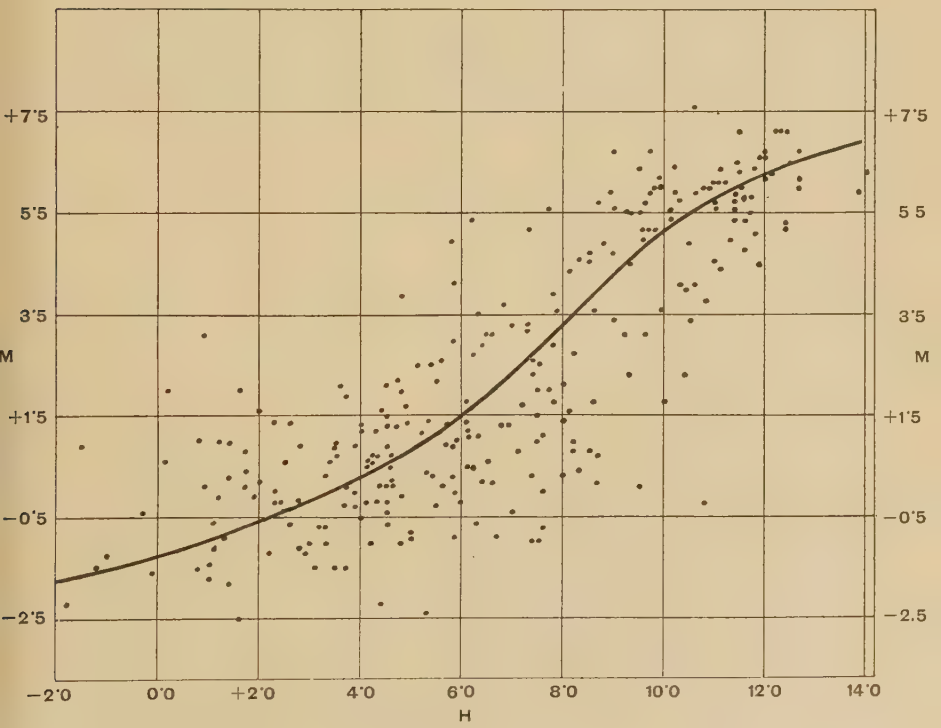


Fig. 2.

There seems little doubt that the gradient is steep in the middle, flattens out considerably for small values of H and possibly for the large values.

Inserting these values in diagram 1, the distribution of the Ko stars in Zone 64° to 72° given in the Draper Catalogue (*i.e.* to a limiting

apparent magnitude of about $8^{m}.5$) according to their absolute magnitude is :—

M.	No. of Stars in Zone.	No. of Stars in Hemisphere.
$<-1^{m}.0$	14	267
$-1^{m}.0$ to $0^{m}.0$	94	1795
$0^{m}.0$,, $1^{m}.0$	182	3476
$1^{m}.0$,, $2^{m}.0$	124	2368
$2^{m}.0$,, $3^{m}.0$	37	707
$3^{m}.0$,, $4^{m}.0$	29	554
$4^{m}.0$,, $5^{m}.0$	13	248
$>5^{m}.0$	9	172

If the limiting apparent magnitude is $8^{m}.5$, then a star whose absolute magnitude is $5^{m}.0$ cannot have a parallax $<''.02$, and must be within a sphere of 50 parsecs. Similarly one whose absolute magnitude is $0^{m}.0$ must be within a sphere of 500 parsecs. Applying this principle to the numbers in the last column, we find in a hemisphere of 50 parsecs :—

0·02 stars with M		$<-1^{m}.0$
0·4	,,	$-1^{m}.0$ to $0^{m}.0$
3·5	,,	$0^{m}.0$,, $1^{m}.0$
9·4	,,	$1^{m}.0$,, $2^{m}.0$
11	,,	$2^{m}.0$,, $3^{m}.0$
35	,,	$3^{m}.0$,, $4^{m}.0$
62	,,	$4^{m}.0$,, $5^{m}.0$
172	,,	$>5^{m}.0$

The 172 would be doubled if the limit of apparent magnitude were $9^{m}.0$.

These figures do not agree with those already obtained. For the bright stars a considerable factor must be allowed for falling off in density at distances like 500 parsecs. But for the stars of greater absolute magnitude the fault probably lies in the scaling from the rough data shown in diagram 2, and weakness of this method. There is agreement in one particular, that the number of stars of magnitudes $1^{m}.0$ to $2^{m}.0$ and $2^{m}.0$ to $3^{m}.0$ are not very unequal.

The conclusion is a very rough one that the " Giants " are most numerous near the magnitude $1^{m}.0$, and that the " Dwarfs " are most numerous near $6^{m}.0$ if we are confined to the magnitude limit of the Draper Catalogue. A few " Giants " apparently extend to about $3^{m}.0$.

Note on the Low Colour-Temperatures of some Early-Type Stars with Strong Ca+ Lines. By W. M. H. Greaves, C. Davidson, and E. Martin.

(*Communicated by the Astronomer Royal.*)

Attention has previously been drawn to the abnormally low colour-temperature of the star ζ Persei, spectral type B1.*

The colour of this star is approximately that of an average star of Type A5, and among stars of early type down to magnitude 3·0 it is conspicuous in this respect.

It is also remarkable in another way. In a paper on "Interstellar Calcium," † O. Struve has given the intensities of the K line of twenty-four early-type stars down to magnitude 3·0. In this list ζ Persei is notable as the star with the most intense K line.

It is natural to inquire whether the two characteristics are related, and an obvious line of investigation is to observe the colour-temperatures of other stars with intense *Ca*+ lines. Struve has drawn attention to five such stars. These stars, together with ζ Persei and with Struve's estimates of the intensities of the K line, are as follows :—

Stars with a Strong K Line.

Star.	R.A. 1900·0.	Dec. 1900·0.	Vis. Mag.	Type.	Int. of K.
	h m	° ′			
ζ Persei	3 47·8	+31 35	2·9	B1	4
α (9) Camelop.	4 44·1	66 10	4·4	Bo	9
χ^2 Orionis	5 58·0	20 8	4·7	B2*p*	13
9 Cephei	21 35·2	61 38	4·9	B2*p*	9
λ Cephei	22 8·1	58 55	5·2	Od	6
Groomb. 3755	22 19·3	54 44	8·4	B2	15

With the exception of the last star these are easily observable with the Greenwich 30-inch reflector with prismatic camera. They have been placed on the observing list, and up to the time of writing, in addition to the observations of ζ Persei already published, four observations of 9 Cephei and four of α Camelopardi have been secured.

Each of these stars has been compared with one of a list of standard stars whose relative colours have been determined on a common basis. These standard stars are being strengthened by further observations, and the results given below have been reduced by adopting values for the gradients of the standard stars from the material so far obtained. These values will be very close to the truth.

The following table gives for several early-type stars, including stars with weaker *Ca*+ lines, the values of the gradient G observed at

* *M.N.R.A.S.*, **88**, 703. H. Kienle has also found a low colour-temperature, *ibid.*, 700.

† *Ap. J.*, **65**, 163, and *Ap. J.*, **67**, 353.

Greenwich. G is connected with the colour-temperature T by the relation

$$G = k - \frac{C_2}{T} \cdot \frac{1}{1 - e^{\frac{-C_2}{\lambda T}}},$$

where $C_2 = 14300$, and λ (measured in microns) is the mean wavelength of the spectral range used in the observations, and may be taken as equal to 0·5. k is a constant adjusted so as to make the mean gradient of A0 stars zero. The values of $\frac{C_2}{T}$, given by Hertzsprung in his list of mean colour equivalents of 734 stars and differenced from 1·48 (his mean value for type A0), are also given.

It may be of interest to mention in this connection that the gradients observed at Greenwich for the standard stars mentioned above have been compared with Hertzsprung's values. From nineteen stars the probable error of the difference Greenwich minus Hertzsprung comes out to be ±0·05, which, if divided equally between the two series, would give a probable error of ±·035 for each. Hertzsprung's average weight for these stars is 140. For some of the fainter stars it is considerably less.

Gradients and Intensities of K.

Star.	Type.	Int. of K. (Struve).	G.	Number of Observations.	$1{\cdot}48-\frac{C_2}{T}$ (Hertzsprung).
γ Pegasi	B2	2	+0·27	7	+0·28
ε Persei	B1	2	+0·22	10	+0·07
η Ursæ Maj.	B3	<1	+0·19	16	+0·19
β Cephei	B1	<1	+0·14	4	+0·23
ζ Persei	B1	4	−0·43	10	−0·33
α (9) Camelop.	B0	9	−0·20	4	−0·27
χ^2 Orionis	B2*p*	13	..	..	−0·92
9 Cephei	B2*p*	9	−0·86	4	−0·57
Mean B3			..		+0·23
Mean A0			0·00		0·00
Mean A5			−0·41		−0·41

It will be seen that the four stars with stronger Ca^+ lines are all redder than an average A0 star. The investigation is to be continued, but from the above values of G taken in conjunction with Hertzsprung's colours it is evident that there is no doubt about this result. The material is slender, but, such as it is, provides ground for suspecting that increased Ca^+ absorption is accompanied by an apparent lowering of the colour-temperatures of early-type stars. The same conclusion has been reached by Struve. An obvious suggestion is that the light is being reddened by passage through an interstellar cloud associated with Ca^+ atoms.

Plaskett has suggested that the calcium clouds are located in the neighbourhood of the early-type stars, but Eddington has preferred the view that the calcium absorption takes place uniformly throughout space, and that stationary *Ca*+ lines * indicate great distance. Reliable parallaxes of the early-type stars, with strong K lines determined at several observatories, would provide a possible check on Eddington's view. The trigonometric parallaxes of B-type stars are as a rule peculiarly liable to systematic error on account of their extreme colour, but this liability will not arise to the same extent in the case of stars such as ζ Persei, whose colour is now found not to be extreme.† It may be mentioned that the parallax of α Camelopardi has been determined at Greenwich to be $+''\cdot023\pm''\cdot007$ from twenty-five plates. If the parallax is really as large as this it would be very unfavourable to Eddington's view. The large value obtained may, however, be due to accidental error, and independent determinations at other observatories are necessary for its verification.

A Wide Absorption Band in some B-type Stars. By William J. S. Lockyer, M.A., Ph.D., Major (late R.A.F.).

In a systematic study of the spectrum of ϕ Persei (Sp. Type Boneη) throughout a complete cycle, a series of photographic enlargements of the spectra was made to illustrate the changes in the spectrum at different phases. These photographs, all enlarged from separate negatives but otherwise treated alike, were mounted consecutively under each other, and it was noticed that a strong region of absorption was apparent in all of them in the same position in the spectrum. This absorption band, while it is an obvious feature in the negatives, is very much more striking in the enlargements.

In a communication ‡ to the Society on the spectrum of this star, this peculiarity in the spectrum was drawn attention to as follows: "A feature conspicuous in all negatives taken at any phase, and more especially noticed in every enlargement that has been made, is that between the ionised iron double at $\lambda\lambda$ 4173·5 and 4178·9, and the ionised iron line at λ 4238·8, this region shows almost continuous homogeneous absorption."

This feature was illustrated in a plate which accompanied the paper, and it was also stated there that it was easily seen in the reproductions of the spectra which accompanied the account of my previous observations § of this star during the years 1924–1925.

To illustrate the prominent nature of this band in many of the

* Struve gives reasons for believing that the lines estimated by him are stationary.

† For example, a gradient of $-0\cdot40$ implies a colour index of $+0\cdot2$.

‡ *M.N.R.A.S.*, **86**, 496, 1926.

§ *M.N.R.A.S.*, **85**, Plates 16 and 17, 1925.

bright-line stars, reference may be made to the plate which accompanied my first paper on the observations of bright-hydrogen-line stars *

As there seemed no reason to expect any absorption in this part of the spectrum, the first explanation suggested for its presence was that it might simply be due to an effect of contrast, indicating only the absence of bright lines, which lines were so prevalent in other parts of the spectrum.

In a more recent communication † to the Society dealing with further observations of the spectra of bright-hydrogen-line stars made during the years 1924 to 1927, attention was again drawn to this feature, which was strongly noticeable in the new series of spectra and apparent also in the spectra reproduced in a new plate. Reference was also made to the absence, so far as I knew, of any origin for the presence of this band.

The new series of spectra suggested, however, that the explanation of the "absence of bright lines" was not tenable, as the band was seen in numerous spectra in which bright lines were very weak or absent throughout the whole spectrum. The possibility that the band might be caused by some local absorption due to the optical parts of the prismatic camera was also considered, but its variable intensity from one spectrum type to another, in cases where the band had an equal opportunity to show itself, led one to look upon the feature as inherent in the spectra themselves.

It is not an easy matter to state exactly the actual width of this absorption band. Thus, for instance, in the case of some stars where the ionised lines of the metals are bright and strong, such as in ϕ Persei (Bo), γ Cassiopeiæ (Bo), v Cygni (B3), and ψ Persei (B5), the band is clearly shown, but the boundaries are sharply defined, and very probably apparently displaced, owing to the proximity of the strong ionised double iron lines $\lambda\lambda$ 4173·5 and 4178·9 near the violet edge, and the ionised single iron line λ 4238·8 near the red edge.

In spectra where the bright ionised lines are not very prominent or where they are entirely absent, the band does not appear so restricted and can be estimated approximately as extending from λ 4170 to λ 4250. This is the case in the following stars:—

P. Cygni, B1	ζ Tauri, B3	κ Draconis, B5
γ Orionis, B2	H.D.C. 20336, B3	β Piscium, B5
c Persei, B3	η Tauri, B5	β Canis Minoris, B8.

There seems little doubt that this absorption band, visible in these hot stars, is identical with that recorded by Dr. Shapley in Vega. Dr. Shapley, in the first ‡ of three papers on this subject, recorded the finding "of a wide absorbing band in the spectrum of Vega, . . . The band extends throughout one-third of the interval between H_γ and H_δ. . . . The suitable spectra of other early type stars were examined and the band found to be universally present, though much fainter in

* *M.N.R.A.S.*, **84**, Plate 9, 1924.
† *M.N.R.A.S.*, **88**, 694, 1928.
‡ *Harvard Coll. Obs. Bull.*, 805, 1924.

some stars, such as Sirius and Altair, . . . Presumably the absorption is to be attributed to the so-called cyanogen band that extends from λ 4215 towards the violet."

In the two other communications,* which were published in March and April of the present year, he analyses the intensity and width of this band for a hundred stellar spectra ranging over all spectrum types.

It is interesting to note that Dr. Shapley also considered (*Bulletin* 856, p. 2) the " possibility that absorption in lens or prisms might give rise to the wide but shallow band," but this explanation he states was eliminated for " a considerable variety of prisms, the 24-inch reflector and several refractors have been used in showing the general prevalence of the absorption band. . . ."

Dr. Shapley clearly points out that he has failed to find the structure of the cyanogen bands in this observed band, and an examination of the numerous Sidmouth photographs, which are of considerable dispersion, also records no trace of them.

It should be noted that the head of the cyanogen band, both in the laboratory and in the sun, is sharply limited at λ 4216. In the spectra of the early or hottest stars photographed at Sidmouth the band extends *at least* as far as λ 4230. For O-type stars Dr. Shapley gives the mean upper limit at about λ 4265.

To account for this displacement towards the red he suggests, as a working hypothesis, that " the source of the cyanogen band is in the immediate vicinity of each star that shows the absorption, the material being cometary or meteoric and not chromospheric or coronal in nature."

From the Sidmouth photographs no opinion can be offered regarding the presence of the cyanogen band at λ 3883 as it falls too near the broad diffuse absorption band of H_ζ. The fainter cyanogen band at λ 4606 lies in a space between bright ionised iron lines, and any absorption appearance could hardly be differentiated from a purely contrast effect.

As regards the presence of these bands in the spectra of early stars, Dr. Shapley is not certain with regard to the former band for the same reason as mentioned above, but in the case of the latter band he states that it certainly cannot be detected.

Summary.

(1) The spectra of many B-type stars photographed at the Norman Lockyer Observatory at Sidmouth display a strong absorption band between the limits λ 4170 and λ 4250 approximately.

(2) The presence of this band cannot be explained as due either to the absence of bright lines in that region or to local absorption caused by the optical parts of the instrument used.

(3) Reference is made to Dr. Shapley's investigation of the presence of this band in all type stars and to his suggested origin as due to cyanogen.

* *Harvard Coll. Obs. Bull.*, 856 and 857.

Stellar Parallaxes Determined at the Royal Observatory, Greenwich.

(Communicated by the Astronomer Royal.)

The following list is a continuation of the results obtained with the 26-inch telescope and published annually in the November number of the *Monthly Notices*.

In the column Proper Motion the first value given is from meridian observations; the second is the relative proper motion derived from the photographs. The motions in both R.A. and Dec. are expressed in arc of a great circle.

In calculating the parallax, the relative proper motion has been used, *not* the meridian proper motion.

In computing the P.E. an average value $\pm''\cdot025$ has been adopted as the probable error of a single plate of unit weight. This value has been derived from a discussion of the residuals of 333 determinations of parallax, each of which depends on 23 plates on the average. The P.E. has accordingly been calculated by the formula

$$r = \frac{\pm''\cdot025}{\sqrt{w}}.$$

The first P.E. given is that of the parallax, the second is the P.E. of the relative proper motion.

Generally six comparison stars of mean magnitude 11–12 have been used, of which good images are given with an exposure of three minutes. The magnitude of the parallax star has been reduced to equality by means of the rotating sector and (for the brighter stars) local desensitisation by means of a weak solution of copper sulphate.

The Visual Magnitude is taken from the *Bonn Durchmusterung*, the Photographic from the *Greenwich Astrographic Catalogue*.

No.	Name and B.D. No.	Mag. Vis.	Mag. Photog.	Spectral Type.	R.A. 1900·0	Dec. 1900·0	Proper Motion. R.A.	Proper Motion. Dec.	Parallax.	P.E.	No. of Plates.
		m	m		h m s		″	″	″	″	
1	66° 34	9·5	11·6		0 25 59	+66° 42′	+1·753	−·198	+·093	±·007	25
							+1·704			±·005	
2	75° 65	9·4	10·8		1 27 0	+75° 29′	+·242	+·043	−·002	±·007	22
							+·280			±·009	
3	Br. 3227	7·8	7·9	Ko	2 7 31	+67° 13′	+·523	−·300	+·035	±·007	20
	67° 191						+·522			±·008	
4	Br. 332	4·7	4·7	A5p	2 20 49	+66° 57′	−·004	+·014	+·019	±·007	24
	66° 213						·000			±·006	

No.	Name and B.D. No.	Mag. Vis. Photog.	Spectral Type.	R.A. Dec. 1900·0.	Proper Motion. R.A.	Proper Motion. Dec.	Parallax.	P.E.	No. of Plates.
		m		h m s	″	″	″	″	
5	66° 268	9·2		3 21 48	+1·185	− ·995	+·016	±·007	22
		10·0		+66° 25′	+1·207	−1·027		±·008	
6	P. III. 121	4·5	Ma	3 40 22	− ·004	+ ·001	−·001	±·007	21
	65° 369	6·1		+65° 13′	+ ·011			±·007	
7	α (9) Camelop.	5·0	Bo	4 44 6	+ ·007	+ ·005	+·020	±·006	25
	66° 358	4·6		+66° 10′	+ ·002			±·005	
8	19 Camelop.	5·2	F8	5 6 4	− ·080	+ ·155	+·056	±·006	26
	79° 169	5·5		+79° 7′	− ·091			±·005	
9	O.A. 6619–20	8·3	K2	6 12 24	+ ·029	− ·106	−·019	±·006	24
	79° 200	9·0		+79° 43′	+ ·058			±·005	
10	O.A. 7011	8·0	K2	6 29 17	+ ·005	− ·100	+·001	±·007	19
	64° 596	8·9		+64° 14′	− ·017			±·009	
11	P. VII. 67	5·5	Ko	7 20 29	− ·003	− ·044	+·006	±·007	20
	68° 480	6·4		+68° 40′	− ·004			±·008	
12	L.F. 1144	6·8	Go	7 36 48	− ·080	− ·176	+·052	±·008	17
	70° 474	7·5		+70° 27′	− ·083			±·008	
13	Gr. 1446	6·4	Ko	8 28 36	− ·017	− ·100	+·011	±·007	23
	74° 370	6·9		+73° 59′	+ ·014			±·006	
14	70° 578	9·5		9 40 17	·000	− ·332	+·021	±·007	22
		10·4		+70° 14′	− ·014			±·005	
15	O.A. 10307	8·9	Ko	9 47 7	− ·075	− ·160	+·019	±·007	23
	65° 743	9·5		+65° 44′	− ·056			±·005	
16	38 Urs. Maj.	5·2	Ko	10 35 8	− ·161	− ·073	+·023	±·007	25
	66° 678	6·3		+66° 14′	− ·174			±·005	
17	λ Draco.	3·3	Ma	11 25 28	− ·038	− ·024	+·021	±·008	23
	70° 665	5·5		+69° 53′	− ·011			±·007	
18	2 Draco.	5·2	Ko	11 30 11	+ ·117	− ·130	+·040	±·007	25
	70° 670	6·1		+69° 53′	+ ·137			±·006	

No.	Name and B.D. No.	Mag. Vis. Photog.	Spectral Type.	R.A. Dec. 1900·0.	Proper Motion. R.A.	Proper Motion. Dec.	Parallax.	P.E.	No. of Plates.
		m		h m s	″	″	″	″	
19	Gr. 2025	6·3	G5	13 34 4	+ ·072	− ·064	+·028	±·008	23
	67° 790	7·4		+67° 33′	+ ·070			±·006	
20	O.A. 14136	8·0	K0	13 53 46	− ·135	+ ·069	+·031	±·008	21
	67° 812	8·8		+67° 26′	− ·145			±·007	
21	5 Urs. Min.	5·0	K2	14 27 44	+ ·012	+ ·017	−·011	±·008	22
	76° 527	5·8		+76° 8′	+ ·020			±·007	
22	C. 2373	8·8	K0	15 39 39	+ ·100	− ·055	+·031	±·009	23
	84° 348	9·0		+84° 50′	+ ·129			±·007	
23	16 Urs. Min.	4·7	A2	15 47 37	+ ·024	− ·003	+·029	±·008	25
	78° 527	4·4		+78° 6′	+ ·036			±·006	
24	15 Draco.	5·2	B8*p*	16 28 11	− ·024	+ ·034	+·030	±·008	26
	69° 850	5·0		+68° 59′	− ·015			±·007	
25	*v* Draco.	5·3	K0	18 55 37	+ ·053	+ ·040	−·021	±·012	20
	71° 915	5·9		+71° 10′	+ ·074			±·008	
26	L. 38677	8·8		19 56 44	− ·052	− ·156	−·013	±·009	25
	79° 650	9·1		+79° 26′	− ·025			±·006	
27	L.F. 3652	6·5	K2	20 56 15	− ·079	− ·045	+·004	±·010	21
	68° 1170	7·7		+68° 40′	− ·079			±·007	
28	C. 3362	9·5		21 52 17	+ ·257	+ ·083	−·012	±·007	38
	85° 368	10·9, 11·9		+85° 27′	+ ·240			±·002	
29	C. 3509	7·3	K0	22 50 6	+ ·098	− ·035	−·029	±·008	22
	84° 513	7·7		+84° 15′	+ ·075			±·008	
30	C. 3659	8·8	K0	23 44 1	+ ·156	+ ·002	−·015	±·008	19
	81° 832	8·6		+82° 14′	+ ·130			±·009	

No. 7. α (9) Camelop. B.D. mag. $5^{m}\cdot0$; Harvard photometry $4^{m}\cdot3$.

A Certain Effect of Asteroidal Perturbation. By T. C. Hudson.

1. In this paper is given a result of an investigation concerning a *selected* portion of all that gross alteration of state of the orbit of an infinitesimal planet which is effected by the long-continued action of one planet Jupiter moving in a fixed ellipse. The *selection* is in respect of a question long asked and long unanswered. One of the main points which emerge, validity, is easily missed. Possibly it is best perceived as a faint comet is best perceived—indirectly.

2. Let us then imagine a child in a railway train, face to engine, travelling as from London towards Edinburgh. In the language of angle any point in the landscape on the Cambridge (or Oxford) side appears to be moving as if from north through east (or west) towards south. Let us suppose its apparent angular velocity at some one instant to be 360 degrees of arc as in one minute of time. Then, "to a first approximation," it might be easy for the child to infer that the same object would again be visible in the same place after the lapse of one minute. Furthermore, if a modification as of the nature of successive approximation were made on the basis of a further supply of angular observations, and if the requisite computing were performed with the aid of any of the well-known and little understood formulæ of finite differences, then it would be the numerical estimate, and the numerical estimate alone, that would be affected by the modification; inferentially speaking it might still seem only a matter of lapse of time for the object to come round again.

3. But the trouble is that it will not come round again. The fact is that the true nature of the apparent velocity is translational. This fact is not to be denied by any non-existence theorem; no arguments whatever, whether they be rational or not, whether they involve integral or fractional, positive or negative powers and products of the expressions "clearly, naturellement, of course," and "kompreneble," will establish the non-existence theorem. But a theorem of this character, that is to say, of argued statement at variance with fact, is still extant, vested as it were in the stereotypes of text-books and memories.

4. In order that we may investigate in such a manner as not to run the risk of making a similar error, if error it be, it is well to adopt some independent method of inquiry which shall involve no validity-question unless it be capable of immediate and satisfactory settlement.

5. Accepting then Newton's Law of Gravitation, and accepting it intact as the sole basis of the logic of the steps of the investigation, we have, first, for a Sun and a Jupiter of mass ratio $1 : M$, a vector differential equation of the second order with respect to time:

$$(d/dt)^2\mathrm{J} + \mathrm{J} \,.\, J^{-3}(1 + M) = 0 \,. \quad . \quad . \quad (5)$$

6. This vector, J, may be either from Jupiter to Sun or else the reverse; say the former.

7. Seeing that M^2 will be neglected, and that the vector J will be multiplied by M when it occurs in the asteroidal equation, we will disregard the factor $1 + M$.

8. The method of analysis adopted in this paper is vectorial.

9. If P and Q are two vectors, their *scalar* product, PQ, is the continued product of their magnitudes, P and Q, and the *cosine* of the angle between them, which cosine is written as pq, which is the scalar product of the directivities, or unit vectors, p and q.

10. But the *vectorial* product, written as VPQ, is *not* an algebraic quantity but a vector. Its directivity is at right angles to P and also at right angles to Q, with a convention as to the one or the other of the two senses. Its magnitude is the continued product of the magnitudes, P and Q, and the *sine* of the angle between them. This sine is written $|V\text{pq}|$. The corresponding vector itself, Vpq, might be called a directed sine, that is a Sine from which the essential property of directivity has not been excised.

11. The former definition, namely that of the scalar product PQ, makes a kind of multiplication possible which reconciles the algebra to the facts of plane geometry of areas (Pythagoras' theorem).

12. The latter definition, that of the vectorial product VPQ, makes another and distinct kind of multiplication possible which reconciles the algebra to the facts of spherical trigonometry.

13. The two definitions, taken together, reconcile the algebra to the facts of the 3-space of our experience.

14. The Jupiter equation of (5) and (7),

$$(d/dt)^2\text{J} + \text{J}J^{-3} = 0 \quad . \quad . \quad . \quad . \quad (14.1)$$

has two vector integral constants, thus

$$d/dt(V\text{J}d\text{J}/dt) = 0 \quad . \quad . \quad . \quad . \quad . \quad (14.2)$$

i.e.

$$V\text{J}d\text{J}/dt = \text{A say} \quad . \quad . \quad . \quad . \quad (14.3)$$

A being invariable,

and

$$d/dt(\text{j} + V\text{A}d\text{J}/dt) = 0 \quad . \quad . \quad . \quad . \quad . \quad (14.4)$$

$$\text{j} + V\text{A}d\text{J}/dt = \text{B say} \quad . \quad . \quad . \quad . \quad (14.5)$$

B being invariable,

j being the directivity or unit vector in J.

15. The vector A may also be written in the forms $\text{a}A$, or $\text{a}|\text{A}|$. The former component, the directivity or unit vector, a, indicates a pole, that is to say a normal to the plane of the orbit, the normal being supposed to be drawn either in the one sense or else in the other sense. The latter or scalar component, the magnitude, A, or $|\text{A}|$, measures area swept out as in two units of time.

16. Next, the scalar product, AB, is $= \text{A}(\text{j} + V\text{A}d\text{J}/dt) = \text{A}\text{j} + \text{A}V\text{A}d\text{J}/dt = 0 + 0 = 0$. Thus AB = 0. The inference is, not at all that $A = 0$ and/or $B = 0$, but simply that ab = 0; that is to

say, b is at right angles to the normal a ; b is in the plane. This directivity or unit vector, b, is readily identifiable as from Sun towards Jupiter-at-his-nearest. The scalar component or magnitude, B, measures the ratio called the eccentricity of the elliptical orbit. The polar equation of a conic follows thus :

$$A^2 = \mathrm{AA} = \mathrm{A}V\mathrm{J}d\mathrm{J}/dt = \mathrm{J}(-V\mathrm{A}d\mathrm{J}/dt) = \mathrm{J}(\mathrm{j} - \mathrm{B}),$$

whence

$$A^2 = J(1 - \mathrm{jb} \,.\, B)$$

which is of the form

$$L = R(1 - \cos\theta \,.\, E).$$

17. The fact of the perpendicularity relation subsisting between A and B, indicates an isolated, scalar, integration (Kepler's equation).

18. The element $d\mathrm{J}$ is describing the closed ellipse again and again. Thus $d\mathrm{J}/dt$ and $V\mathrm{A}d\mathrm{J}/dt$ are wholly periodic ; they have no temporal mean value or constant part. Let us agree to signify this state of affairs with the aid of the symbolism " $\pm$ . . . ", thus,

$$V\mathrm{A}d\mathrm{J}/dt = 0 \pm \ldots \quad . \quad . \quad . \quad . \quad (18.1)$$

Hence

$$\mathrm{j} = \mathrm{B} \pm \ldots \quad . \quad . \quad . \quad . \quad (18.2)$$

that is to say, B, the vector of the perihelion and the eccentricity, is identical with the mean temporal value, or constant part, of the inward directivity j, drawn as from planet towards Sun. And again,

$$\mathrm{bj} = B \pm \ldots \quad . \quad . \quad . \quad . \quad (18.3)$$

that is to say, the eccentricity B is identical with the omnitemporal mean value of all the cosines defined as bj, that is of the cosines of the longitudes from aphelion. On the other hand, the corresponding *sine* is wholly periodic, for we have

$$\mathrm{j}V\mathrm{ab} = 0 \pm \ldots \quad . \quad . \quad . \quad . \quad (18.4)$$

19. The mean value of J, the vector itself, is known to be

$$\mathrm{B} \,.\, A^2(1 - B^2)^{-1\frac{3}{2}}. \quad . \quad . \quad . \quad . \quad (19)$$

20. Again, if $\mathrm{jb} = -C$ and $\mathrm{j}V\mathrm{ab} = -S$ in which $C^2 + S^2 = 1$, and if $(1 + BC)(1 - B\gamma) = 1 - B^2$, and if $\mathrm{b}\gamma + V\mathrm{ab}\sigma = -\mathrm{p}$, and $\gamma^2 + \sigma^2 = 1$, then $-\mathrm{p}$ is the directivity or unit vector which defines the moving arm of the eccentric anomaly, if $S\sigma$ is positive.

21. We have also, if i and k are integers not negative,

$$(1 + B\gamma)\gamma^{2i+1}\sigma^k = 0 \pm \ldots \quad . \quad . \quad . \quad (21)$$

22. If $x\,!$ denote the continued product of the integers 1, 2, 3, . . . $x - 1$, x we know that

$$\frac{i\,!\,(2\gamma)^{2i}}{2i\,!}\,\frac{k\,!\,(2\sigma)^{2k}}{2k\,!}(i + k)\,! \pm \ldots = 1 \quad . \quad . \quad (22.1)$$

and $$\gamma^i\sigma^{2k+1} \pm \ldots = 0 \quad . \quad . \quad . \quad . (22.2)$$

and if eccentricity be not regarded,

$$\gamma^{2i+1}\sigma^k \pm \ldots = 0 \quad . \quad . \quad . \quad . (22.3)$$

23. These considerations will suffice for our purpose, in the elimination of the Jovian time-variable, when we come to deal with the asteroid.

24. We now require the vectorial equation of Newtonian motion of the asteroid itself; mass o. It is

$$(d/dt)^2\mathrm{R} + \mathrm{R}R^{-3} = (\mathrm{J}J^{-3} + \mathrm{D}D^{-3})M \quad . \quad . (24.1)$$

in which

$$\mathrm{J} + \mathrm{R} + \mathrm{D} = 0 \quad . \quad . \quad . \quad . (24.2)$$

J being from Jupiter to Sun, R from Sun to asteroid, D from asteroid to Jupiter.

25. Let us denote

$$V\mathrm{R}d\mathrm{R}/dt \text{ by } \mathrm{A}' \quad . \quad . \quad . \quad . (25.1)$$

and

$$V\mathrm{A}'d\mathrm{R}/dt + \mathrm{r} \text{ by } -\mathrm{B}' \quad . \quad . \quad . (25.2)$$

Then A′ and B′ are analogous to A and B.

26. These vector integrals A and B would be constant if it were not for M, the measure of the perturbing mass. They might be called, agreeably with custom, vector "variable constants"; A′ might be called the "oblique" variable and B′ might be called the eccentric variable. When they occur with the factor M they may here be treated as constants because we are neglecting M^2.

27. Writing

$$d\tau \text{ for } Mdt \quad . \quad . \quad . \quad . (27.1)$$

one finds

$$d\mathrm{A}'/d\tau = V\mathrm{RJ}(J^{-3} - D^{-3}) \quad . \quad . \quad . (27.2)$$

and

$$d\mathrm{B}'/d\tau \pm \ldots = V\mathrm{A}'(2 - \mathrm{r}\,.\,\mathrm{r})(\mathrm{R} + \mathrm{J})D^{-3} + [\ ] \quad . (27.3)$$

in which

$$[\ ] = \{(V\mathrm{A}' - Vd\mathrm{R}/dtV\mathrm{R})d/dt\}d\mathrm{J}/dt \quad . \quad . (27.4)$$

and is seen to be negligible, as wholly periodic, if we stipulate incommensurability of the Jovian and asteroidal periods.

28. It is now convenient to state what it is that we are *selecting*. Writing $d\mathrm{B}'/dt$, the temporal variation of B′ the vector of asteroidal perihelion and eccentricity, in the form $MB\mathrm{X}$, thus

$$d\mathrm{B}'/dt = MB\mathrm{X} \quad . \quad . \quad . \quad . (28)$$

we are selecting that part of X which is independent of (1) the time, (2) the mass measure M, (3) B the eccentricity of Jupiter's orbit, (4) B' the eccentricity of the orbit of the asteroid itself. Let the symbolism "$\pm \ldots$" now indicate this selection.

29. Let $A^2 = K\xi$, $R \pm \ldots = (A')^2 = K\eta$, $\xi^2 + \eta^2 = 1$, $2\xi\eta = \theta^{\frac{1}{2}}$, $\mathrm{p}\xi + \mathrm{r}\eta = \delta$ (a vector)

$$-\mathrm{J} \pm \ldots = (-\mathrm{B} - \mathrm{p})K\xi$$
$$-\mathrm{p} = (\gamma + V\mathrm{a}\sigma)\mathrm{b}$$
$$\gamma^2 + \sigma^2 = 1$$

$\mathrm{N} = V\mathrm{aa}'$ (*directed* Sine of the inclination, along the line of nodes),

$r = (C + V\mathrm{a}'S)\mathrm{n}$
$C^2 + S^2 = 1$, C and S not being the C and S of § 20.

30. Then one finds that

$$dB'/d\tau \pm \ldots = K^{-2}BA'(-\mathrm{Y}\xi + [\ ]) \quad . \quad . \quad (30.1)$$

in which

$$\mathrm{Y}\xi = V\mathrm{a}'(2 - \mathrm{r}\,.\,\mathrm{r})\{(\delta\gamma + 2\mathrm{b}\xi)|\delta|^{-3} + 3\xi(V\delta)^2\mathrm{b}|\delta|^{-5}\} \quad . \quad (30.2)$$

and $[\ ] = (1 + B\gamma)V\mathrm{a}'(2 - \mathrm{r}\,.\,\mathrm{r})\delta|\delta|^{-3}$ and is negligible by (21).

31. In detaching the non-periodic parts it is found, after substituting $1 - \xi^2$ for η^2 where called for, that all the effective terms are characterised by the factors $\xi\theta^i$, $i = 1, 2, \ldots \infty$. One sees then why it is that the first few terms vanish.

32. Proceeding, one finds

$$dB'/d\tau \pm \ldots = K^{-2}BA'\xi Z \quad . \quad . \quad . \quad (32.1)$$

in which

$$Z = \Sigma V\mathrm{a}'\mathrm{r}\,.\,\mathrm{rb}\frac{\theta^i(4i+1)!\,i}{2^{bi}(2i)!\,(i+1)!\,i!}\lambda_i \quad . \quad . \quad (32.2)$$

in which

$$\lambda_i = -((V\mathrm{ar})^2)^i(2\iota + 3)$$
$$+ ((V\mathrm{ar})^2)^{i-1}(2\iota + 2) \quad . \quad . \quad . \quad . \quad (32.3)$$

Thus the time-variable pertaining to Jupiter and implicit in γ and σ has now disappeared.

33. At this stage it is convenient to contemplate the simplified case, of co-planarity. We may then substitute 1 for $(V\mathrm{ar})^2$, and -1 for λ_i. Then the sole representative of the asteroid remaining in the expression for $Z \pm \ldots$ is $V\mathrm{a}'\mathrm{r}\,.\,\mathrm{r}$. Its mean value (since we neglect the eccentricity of the orbit of the asteroid itself) is not affected by shifting the epoch through a right angle. Thus $V\mathrm{a}'\mathrm{r}\,.\,\mathrm{r} \pm \ldots = V\mathrm{a}'V\mathrm{a}'\mathrm{r}\,.\,V\mathrm{a}'\mathrm{r} \pm \ldots =$ half the sum $\pm \ldots = \frac{1}{2}V\mathrm{a}'$.

34. Hence (32) becomes

$$dB'/d\tau \pm \ldots = K^{-2}BA'\xi Z \text{ in which } Z$$

$$= +\tfrac{1}{2}V\mathrm{ba}'\Sigma\frac{\theta^i(4i+1)!\,i}{2^{bi}(2i)!\,(i+1)!\,i!}$$

$$= \tfrac{3}{2}V\mathrm{ba}'\xi\eta\Sigma\frac{(\xi\eta)^{2i-1}(4i+1)!}{3\,.\,2^{2i}(2i)!\,(2i-1)!}\,\frac{2i!\,2!}{2^{2i+2}(i+1)!\,i!}$$

$$= \pm \ldots + \tfrac{3}{2}V\mathrm{ba}'\xi\eta CS^2(1 - 2\xi\eta C)^{-\frac{5}{2}} \quad . \quad . \quad . \quad (34.1)$$

whence $dB'/d\tau \pm \ldots$ becomes

$$K^{-3}VJA' \,.\, \xi\eta CS^2(1 - 2\xi\eta C)^{-\frac{5}{2}} \pm \ldots$$

35. Now the non-periodic part of this term cannot vanish. Nor can it cancel out with terms subsequently obtained in further approximation in which account is taken of higher orders of Jovian mass and Jovian eccentricity. And it is independent of the cyclic term (not here considered) which depends on the eccentricity of the orbit of the asteroid itself. It is true that in a solar system the near vertex of the orbit of a Jupiter may have a mean motion ; but its secondary effect on the secular part of dB'/dt, the variation of the vector of asteroidal perihelion and eccentricity (and, therefore, also incidentally on the secular motion of any representative point on the major axis, on the assumption of the secular invariance of the asteroidal period), is to transmute linear into spiral, not linear into circular.

36. The logic here attempted, whether good and severe or not, is based on the Newtonian Law, solely, without canonical transformation. If good and severe, it evidences a conditional precertitude as to ultimate shock and it indicates a generalisation of far-reaching consequence.

Micrometrical Measures of Potsdam Double Stars, made with the 10-*in. Refractor of the Armagh Observatory.* By Mervyn A. Ellison.

(*Communicated by the Director, Armagh Observatory.*)

The following list is the first of a series of remeasures of the Scheiner doubles constituting the Potsdam Catalogue.* At the outset it was decided only to measure pairs under 10″ since those of greater separation can be dealt with more effectively by photography. A number have been measured by Espin and Jonckheere at various times, while some are identical with pairs discovered by these observers. The majority, however, have been neglected since the beginning of the century when the Scheiner Catalogue was compiled from the Potsdam plates.

The magnitudes given by Scheiner are photographic, and in many cases only approximate, especially when the components are faint. Visual estimates have been substituted and a number of other corrections and later information added. Finally, all the places have been brought up to 1930.

* *Katalog von Doppelsternen der Photographischen Himmelskarte aus der Zone von* +31° *bis* +40° *Deklination*, J. Scheiner. Publikationen des Astrophysikalischen Observatoriums zu Potsdam, Nr. 59, 1908.

Scheiner No.	B.D.	1930. R.A.	1930. Dec.	P.	D.	Mags.	Nights.	Date 1920.	
	°	h m	° ′	°	″			+	
4	+31·50	0 21·6	+32 8	167·0	5·15	8·8, 9·0	2	7·661	
8	..	0 37·2	+32 34	271·3	5·05	10·8, 11·0	3	7·870	
12	+31·175	1 1·5	+31 42	61·9	4·84	9·8, 10·3	3	7·872	
16	..	1 13·7	+32 25	175·7	5·72	10·3, 10·5	2	7·985	
20	..	1 48·6	+31 7	104·9	7·45	9·3, 10·4	2	7·983	
21	..	1 58·6	+32 52	81·8	9·13	10·9, 11·3	2	8·304	
26	+31·527	2 57·2	+32 9	7·7	8·33	8·4, 8·8	2	7·987	
27	..	3 1·5	+31 47	32·6	4·25	10·7, 11·0	2	7·987	
40	+31·759	4 15·8	+31 40	108·6	9·79	9·7, 9·8	3	8·089	
42	+32·771	4 16·7	+32 40	267·5	9·67	9·1, 11·3	2	8·305	
515	+32·1892	9 28·1	+31 57	175·9	5·15	9·4, 10·1	2	8·258	
532	+32·2255	12 29·9	+32 26	341·4	7·82	8·3, 8·5	2	8·245	
547	..	17 43·4	+32 8	245·0	8·86	10·0, 11·0	3	7·536	
550	..	17 51·7	+31 36	39·9	6·70	10·0, 11·0	3	7·554	
553	..	17 54·4	+32 53	27·1	9·39	10·0, 10·2	2	7·535	
570	..	18 39·4	+35 0	130·5	8·83	9·9, 11·3	2	7·555	
577	+32·3358	19 10·0	+32 40	12·3	9·52	8·8, 11·0	2	7·555	
587	+38·3512	19 14·1	+39 3	288·7	3·71	8·3, 12·1	2	7·583	
590	+38·3547	19 18·2	+39 3	301·3	9·51	8·5, 9·5	2	7·535	
595	..	19 21·6	+34 19	280·2	7·77	11·3, 11·7	2	7·590	
602	+39·3765	19 24·1	+39 25	338·3	3·83	9·5, 10·0	2	7·535	
612	..	19 28·2	+34 30	29·3	8·03	10·9, 11·3	2	7·590	
628	+39·3813	19 30·8	+39 14	331·4	9·02	9·0, 10·5	2	7·537	
632	..	19 31·0	+38 36	258·0	8·10	10·0, 11·0	2	7·560	
634	+32·3478	19 31·2	+32 40	223·5	8·41	8·8, 9·0	2	7·535	
667	+34·3682	19 40·9	+34 40	156·9	4·54	9·1, 9·4	2	7·560	
682	+39·3912	19 45·2	+39 46	65·6	4·37	10·0, 11·1	2	7·560	AB
683	..	19 45·2	+39 46	20·7	18·29	C= 11·0	1	7·666	AC
691	+39·3925	19 47·2	+39 28	153·5	7·47	9·0, 11·3	2	7·618	
692	..	19 47·3	+33 57	12·1	2·97	10·4, 10·5	2	7·661	
695	..	19 48·0	+34 19	339·1	7·91	9·8, 10·5	2	7·657	
697	..	19 48·4	+34 33	309·7	4·02	10·7, 11·1	2	7·669	
699	..	19 48·6	+34 24	243·3	4·03	11·3, 11·5	2	7·693	AB
700	..	19 48·6	+34 24	177·9	26·98	C= 11·0	1	7·707	AC
705	..	19 49·2	+34 51	131·7	9·28	10·0, 10·5	2	7·657	
708	+33·3628	19 49·6	+33 39	289·4	8·12	9·2, 11·0	2	7·669	
717	..	19 51·1	+38 48	105·8	4·05	10·8, 11·0	2	7·687	
723	..	19 52·0	+34 42	25·9	7·05	10·1, 10·9	2	7·669	
724	+36·3764	19 52·3	+36 34	292·1	7·95	9·8, 10·7	2	7·669	

Scheiner No.	B.D.	1930. R.A.	1930. Dec.	P.	D.	Mags.	Nights.	Date 1920. +	
	°	h m	° ′	°	″				
726	+36·3767	19 52·5	+36 42	15·4	7·56	9·9, 10·7	2	7·669	
727	+34·3791	19 52·6	+36 24	229·0	4·39	10·0, 10·0	3	5·877	
				229·2	4·58	10·0, 10·0	1	7·666	
740	..	19 53·9	+39 24	164·8	8·71	10·7, 11·1	2	8·629	
742	..	19 54·5	+39 11	143·5	7·47	10·7, 11·1	2	8·449	
745	+34·3808	19 54·9	+34 43	272·1	6·97	9·8, 9·9	3	5·954	
				271·3	7·28	9·3, 9·5	2	8·629	
748	..	19 55·2	+34 57	29·7	5·81	10·3, 10·6	2	8·629	
750	..	19 55·3	+38 31	22·9	6·41	10·5, 10·9	2	8·632	
753	..	19 55·5	+33 35	141·3	6·35	10·8, 11·2	2	8·632	
755	..	19 56·0	+38 19	71·3	3·65	10·3, 10·7	2	8·679	
759	..	19 56·5	+35 29	52·9	5·74	10·8, 11·0	3	8·679	AB
				50·6	17·27	C= 12·5	2	8·697	AC
760	..	19 56·5	+37 48	352·6	5·85	10·4, 10·7	2	8·669	
765	..	19 56·6	+36 51	93·3	8·51	11·0, 11·0	2	8·669	AB
				197·9	21·47	C= 11·5	1	8·695	AC
770	+35·3902	19 56·8	+35 43	326·5	7·53	8·4, 10·6	2	8·669	
1529	+35·4599	21 36·0	+36 4	159·0	7·75	8·9, 10·1	2	8·669	
1531	..	21 37·3	+36 44	20·2	2·97	10·5, 10·5	2	8·669	
1537	+34·4542	21 47·1	+34 30	257·1	9·47	8·5, 10·3	2	8·671	
1541	+39·4695	21 49·4	+40 6	164·6	4·98	10·3, 10·5	2	8·702	
1543	+39·4703	21 50·7	+39 33	82·5	8·51	8·5, 10·4	2	8·702	
1545	..	21 54·2	+38 37	324·5	5·33	10·0, 10·5	2	8·702	
1553	+33·4417	22 1·1	+34 13	32·9	5·70	9·0, 11·0	2	8·702	
1555	+37·4479	22 1·3	+37 44	177·9	7·23	11·0, 11·0	2	8·702	
1564	+37·4898	23 48·3	+37 29	87·9	5·21	8·8, 9·0	2	8·702	

Notes.

4. Σ 29. Fixed.
12. This is B.D. + 31°·175, not + 31°·75 as given in the Potsdam Catalogue. Angle increasing, distance decreasing.
16. This is H 645.
26. Persei 104, Σ 336, B.D. + 31°·527, not + 31°·526.
27. This is Espin 326 BC. It is *2′ nf* B.D. + 31°·536.
40. This is H 5460. Not given as a B.D. star in β's G.C. or by Scheiner.
532. This is Σ 1653. Fixed.
547. B is probably less than 11·0 mag.
587. A 158. Apparently no change.
590. This is Σ 3131. β says fixed, but angle seems to be decreasing slowly.
595. *f* star is the brighter to me. Scheiner P. 100°·4, D. 7″·98.
602. This is J 1157. Both Espin and Jonckheere take B as the brighter, but I find Scheiner's mags. nearer the truth.
634. This is H 1413. Angle probably increasing.
667. A.G. 236. Probably no change. Scheiner's angle too large? β's measure (1903·5) agrees with mine.

682. Angle decreasing. AC is Scheiner 683.
691. This is H 2903.
699. 3′ *sf* B.D. + 34°·3754. Large change in angle from Scheiner's measure (1894·6), P. 330°·6, D. 3″·22. Possibly a misprint for 230°·6 ? AC is Scheiner 700.
717. Probably no change. Measured by Espin (1923), who also added a *comes*, 14·0 mag., P. 318°·3, D. 21″·6. This was not seen in the 10-inch.
723. Increasing angle and distance.
727. This is Espin 200, also Holmes 32. All the measures show it to be fixed.
745. Holmes 35.
750. Measures of this pair are rather discrepant in angle. Espin 1923·708, P. 28°·9, D. 6″·39, mags. 10·1, 11. Scheiner gives two measures of angle, P. 26°·0 and P. 21°·0.
753. 1′ *s* is Ellison 8. P. 86°·9, D. 3″·87, mags. 10·8, 11·3, 1928·669.
755. Increase in distance and decrease in angle from Scheiner's measure.
759. This is not a B.D. star as given in the Potsdam Catalogue. B is probably the brighter. *Comes*, 12·5 mag. added with the 10-inch.
765. *Comes*, 11·5 mag. added with the 10-inch.
1529. This is Σ 2814. Probably fixed.
1531. This is J 1146. Jonckheere takes B as the brighter.
1537. This is H 1697.
1541. This is Espin 261 (P. 162°·5, 1905·96). Scheiner's angle is evidently too small, but there may be a slow increase.
1543. Scheiner's position is nearly 1 m. too large in R.A.
1555. The *n*-star is the brighter to me, also Espin (1917).
1564. Σ 3042. Fixed.

Missing Doubles.

11. $0^h\ 55{\cdot}6^m$, + 31° 39′·4, 1927 Dec. 27. Not found.
517. $9^h\ 49{\cdot}1^m$, + 31° 7′·2, B.D. + 31°·2061. This star was measured by me, 1928·276, P. 347°·1, D. 28″·19, mags. 9·2, 10·2. Espin finds it to be Oxford + 31°·24291, 24298, which give P. 345°·5, D. 27″·6, mags. 10·6, 11·8, 1903 April 18. There is no pair resembling Scheiner 517. His D. 7″·80 might be a misprint for 27″·8, but the angle does not agree.
653. $19^h\ 35{\cdot}5^m$ + 36° 0′·5. Not found 1927 Aug. 8 and 29.
767. $19^h\ 56{\cdot}6^m$ + 38° 20′·1, B.D. + 38°·3848. No *comes* found by Espin (1923) or myself.
1561. $22^h\ 13{\cdot}5^m$ + 36° 47′·0. Scheiner's mag. 7·5 is doubtful. No pair found, several searches.

1928 *September* 17.

Observation of Meteors from Skjellerup's Comet, 1927 k.

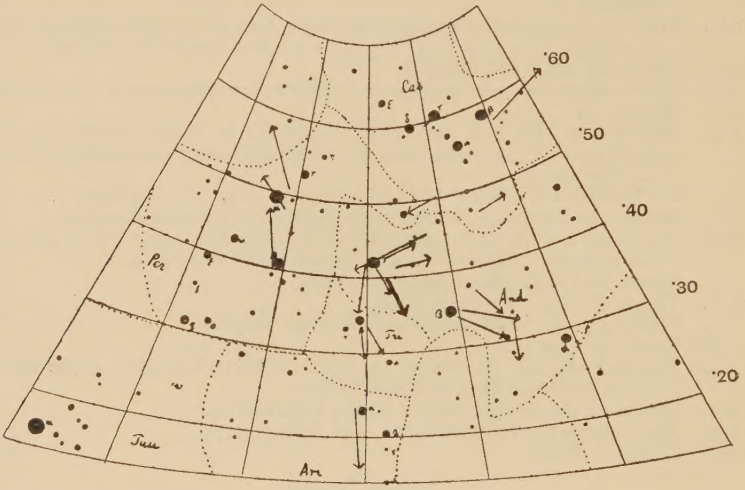
By Kaname Nakamura.

(*Communicated by Professor I. Yamamoto.*)

The possibility of the appearance of a new meteoric shower from Skjellerup's comet, 1927 *k*, was pointed out by Professor Yamamoto of our observatory,* and my observation was accordingly undertaken. Unfortunately the weather conditions in the first decade of June were quite unfavourable and only one morning was available for observation. It was the morning of June 8 that had cleared up, but the moon, aged twenty days, was bright, and the predicted radiant point was low above the horizon. The watch was made during an interval of thirty minutes

* *M.N.* (1928 March), p. 487.

from 2.40 to 3.10, Japanese Central Standard Time, and nineteen meteors were recorded. The following table describes the meteors recorded in the accompanying map.



List of Meteors observed on 1928 *June* 8.

No.	Time. h m	Appearance. ° °	Disappearance. ° °	Mag.
1*	2 40	15+34	9 +31	5
2	2 41	27+40	24 +35	5
3*	2 44	27+42·5	22·5+44	5
4	2 45	30+42	31 +35	5
5	2 47	46+40	49 +48	5
6	2 48	16+50	24 +48	5
7	2 49	29+41	26 +38	5
8	2 49	7+32·5	7 +27·5	5
9*	2 50	47+48·5	53 +51	5
10*	2 52	9+37·5	3 +38	5
11*	2 55	31+34	30 +29	5
12	2 56	11+37	8 +33	5
13*	2 56	15+35	6 +32	4
14*	3 00	25+41	19 +41·5	5
15*	3 01	359+57·5	346 +59	4
16	3 02	17+45	32 +41	5
17*	3 05	47+51	54 +58	5
18*	3 07	32+23	31 +27	5
19*	3 09	30+34	27·5+30	5

About eleven meteors seem to have radiated from 34°+38°, which position is quite near the revised radiant computed by Drs. Crommelin and Yamamoto.* The identity seems to be probable in spite of the general faint character of the meteors, but careful watch in the coming years will be required for definite confirmation.

* *M.N.* (1928 May), p. 596.

Kyoto University Observatory.